Mathematatic

First Stage

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Doctor

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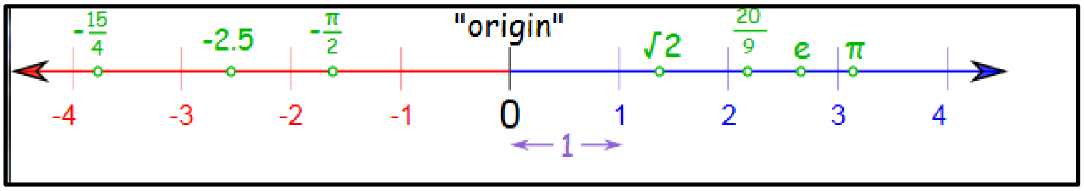
The Real Number Line

The real numbers (Consisting of all integers, fractions, rational and irrational numbers) can be represented as a line , called real number line (Figure 1) .



We distinguish three special subsets of real number

1- Integer , namely , 0 ,± 1 , ± 2 , ---------
2- Natural numbers , namely 1,2,3,4,-----------
3- Irrational numbers, example π ,$\sqrt{2}$,$\sqrt[3]{2}$ and $\log_{10} 3$
4- The rational numbers , namely the numbers that can be expressed in the form of a fraction $\frac{m}{n}$, where m and n are integers and n ≠ 0 .Examples are $\frac{1}{3}$, $-\frac{4}{9} = \frac{4}{-9}$, $\frac{200}{13}$ $and\ 57 = \frac{57}{1}$

The rational numbers are precisely the real numbers with decimal expansions that are either

(a) Terminating (ending in an infinite string of zeros), for example

$$\frac{3}{4} = 0.750000000 \ldots \ldots = 0.75$$

(b) Eventually repeating (ending with a block of digital that repeats over and over), for example

$\frac{23}{11} = 2.09090909 \ldots \ldots = 2.\overline{09}$ the bar indicates the block of repeating block of repeating digits .

Set notation is very useful for specifying a particular subset of real numbers. A set is a collection of objects, and these objects are the elements of the set . If S is a set , the notation a∈S means that a is an element of S, and a∉S means that a is not an element of S . If S and T are sets, then S ∪ T is their union and consists of all elements belonging either to S or T . The intersection S ∩ T consists of all elements belonging to both S and T .The empty set ϕ is the set that contains no elements . For example , the intersection of the rational numbers and the irrational number is the empty set. If set A consisting of the natural numbers (or positive integers) less than 6 can be expressed as

$$\boldsymbol{A} = \{\, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$$

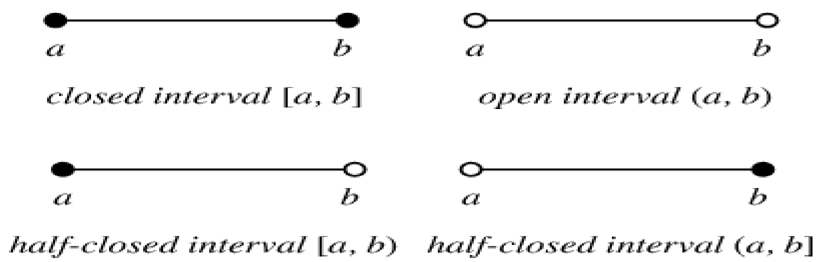
The entire set of integers is written as

$$\{\mathbf{0}, \pm\mathbf{1}, \pm\mathbf{2}, \pm\mathbf{3}, \dots \dots\}$$

Another way to describe a set is to enclose in braces a rule that generates all the elements of the set. For instance, the set

$$A = \{x | x \text{ } is \text{ } an \text{ } integer \text{ } and \text{ } 0 < x < 6\}$$

Interval



An interval is a connected portion of the real line.

1. If the endpoints a and b are finite and are included, the interval is called closed and is denoted $[a, b]$.
2. If the endpoints are not included, the interval is called open and denoted (a, b). If one endpoint is included but not the other, the interval is denoted $[a, b)$ or $(a, b]$ and is called a half-closed (or half-open interval).
3. An interval $[a, a]$ is called a degenerate interval.
4. If one of the endpoints is $\pm\infty$, then the interval still contains all of its limit points, so $[a, \infty)$ and $(-\infty, b]$ are also closed intervals. Intervals involving infinity are also called rays or half-lines.

Notation	Set description	Type	Picture
(a,∞)	{x | x> a}	Open	 a
[a,∞)	{x | x≥ a}	Closed	 a
(-∞ , b)	{x | x< b}	Open	 b
(-∞ , b]	{x | x≤ b}	Closed	 b
(- ∞ ,∞)	R(set of all real number)	Both open and close	

Absolute value

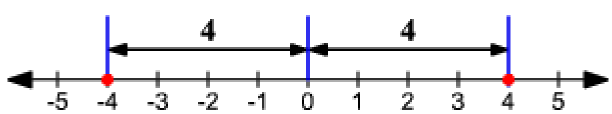
<u>The **absolute value**</u> of a number, denoted $|x|$ is its distance from zero on a number line. For instance, 4 and −4 have the same absolute value (4).

Example 1

Evaluate the following expressions.

1. $|-7| = 7$ **2.** $|3| = 3$ **3.** $|0| = 0$

4. $\left|\frac{2}{3}\right| = \frac{2}{3}$ **5.** $|-3.7| = 3.7$ **6.** $-|-6| = -6$



Example of absolute value

$\mathbf{1.}\ |-7| = 7$ $\mathbf{2.}\ |3| = 3$ $\mathbf{3.}\ |0| = 0$ $\mathbf{4.}\ -|-6| = 6$

$\mathbf{5.}\ \left|-\frac{2}{3}\right| = \frac{2}{3}$ $\mathbf{6.}\ |-3.7| = 3.7$

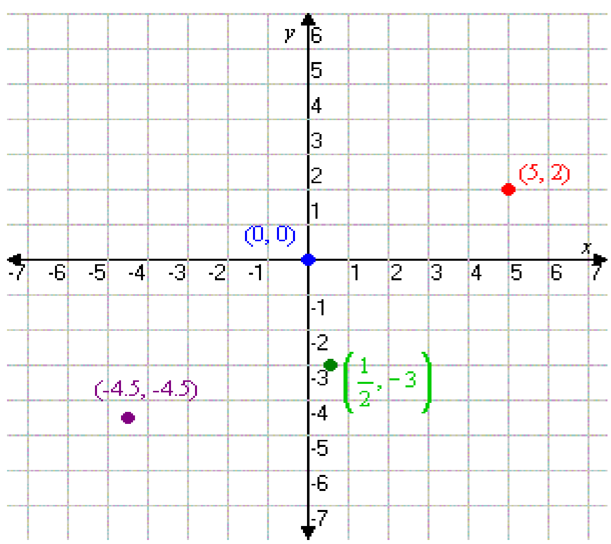
The absolute value parent function, written as $\boldsymbol{f(x) = |\,x\,|}$ is defined as

$$f(x) = \begin{cases} x \quad if\ x \geq 0 \\ \\ -x\ if\ x < 0 \end{cases}$$

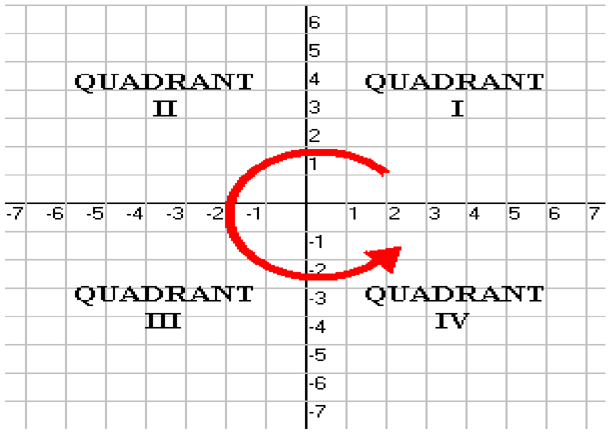
1- If the number **a** is positive or zero (≥ 0) , if the number enclosed in the absolute bars is a non – negative number , the absolute value of the number is the number itself
2- If the number **a** is negative (< 0), if the number enclosed in within the absolute value bars is opposite of a negative number is positive number.

The Cartesian Plane

A Cartesian plane is defined by two perpendicular number lines: the x-**axis**, which is horizontal, and the y-**axis**, which is vertical. Using these axes, we can describe any point in the plane using an ordered pair of numbers. The Cartesian plane extends infinitely in all directions. The location of a point in the plane is given by its coordinates, a pair of numbers enclosed in parentheses: (x, y). the first number x gives the point's horizontal position and the second number y gives its vertical position. All positions are measured relative to a "central" point called the origin, whose coordinates are (0,0).



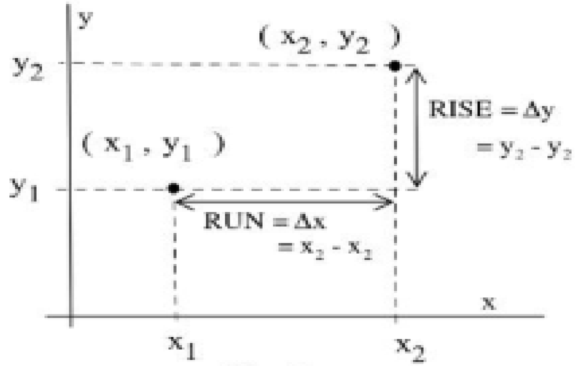
The Cartesian plane is divided into four **quadrants**. These are numbered from I through IV, starting with the upper right and going around counterclockwise. (For some reason everybody uses roman numerals for this).



In Quadrant I, both the x- and y-coordinates are positive; in Quadrant II, the x-coordinate is negative, but the y-coordinate is positive; in Quadrant III both are negative; and in Quadrant IV x is positive but y is negative. Points which lie on an axis (i.e., which have at least one coordinate equal to 0) are said not to be in any quadrant. Coordinates of the form (x,0) lie on the horizontal x-axis, and coordinates of the form (0,y) lie on the vertical y-axis.

Increments and distance between points in the plane

If we move from a point P=(x_1 , y_1) to a point Q = P=(x_2 , y_2) in the plane , then we will have two increments or changes to consider . The increment in the x or horizontal direction is $x_2 – x_1$ which is denoted by $\Delta x = x_2 - x_1$. The increment in the y or vertical direction is $\Delta y = y_2 - y_1$. These increments are shown in figure below. Δx dose not represent Δ times x , it represents the difference in the x coordinates: $\Delta x = x_2 - x_1$.

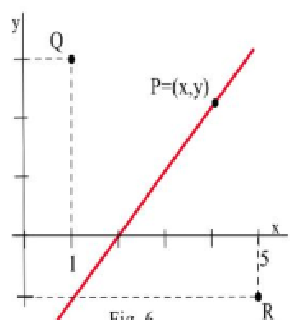


The distance between the points P=(x_1 , y_1) and Q = P=(x_2 , y_2) is simply an application of the Pythagorean formula for right triangles and

$$dist(P,Q) = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The midpoint M of the line segment joining P and Q is $M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$

Example: - Find an equation describing the points P = (x,y) which are equidistant from Q=(2,3) and R=(5,-1) .



Solution: - The points P=(x ,y) must satisfy from Q=(1,3) and R=(5,-1) so

$$= \sqrt{(x-1)^2 + (y-3)^2} = \sqrt{(x-5)^2 + (-1)^2}$$

By squaring each side we get $= (x-1)^2 + (y-3)^2 = (x-5)^2 + (y+1)^2$

$$= x^2 - 2x + 1 + y^2 - 6y + 9 = x^2 - 10x + 25 + y^2 + 2y + 1$$

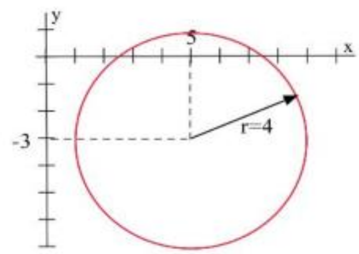
$$= -2x - 6y + 10 = -10x + 2y + 26$$

$$y = x - 2\ , a\ straight\ line\ every\ point\ on\ the\ line\ y = x - 2\ is\ equally\ distant\ from\ Q\ and\ R$$

Practice:- Find equation describing all points P=(x,y) equidistant from Q=(1,-4) and R=(0, - 3) .

Note:- A circle with radius r and center at the point C=(a,b) consists of all points P=(x ,y) which are at a distance of r from the center C:the points P which satisfy dist(P,C)=r .

Example: - Find the equation of circle with radius r=4 and center C=(5 , –3)



Solution: - A circle is the set of points P=(x,y) which are at a fixed distance r from the center point C, so this circle will be the set of points P=(x,y) which are at a distance of 4 units from the point C=(5 , - 3) .P will be on this circle if dist(P,C)=4 using the distance formula and simplifying .

$$\sqrt{(x-5)^2 + (y+3)^2} = 4\ \ so\ \ (x-5)^2 + (y+3)^2 = 16$$

$$or$$

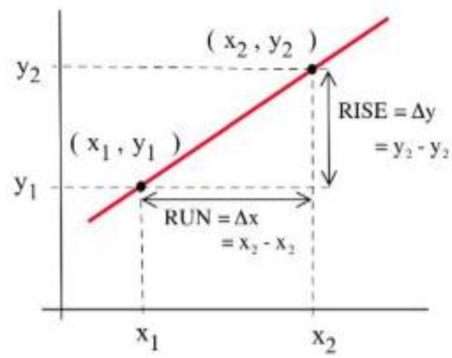
$$x^2 - 10x + 25 + y^2 + 6y + 9 = 16$$

Example: - Find the equation of circle with radius r=4 and center C=(-2 , 6)

The slope between points in the plane

In one dimension on the number line, our only choice was to move in the positive direction (so the x- value were increasing) or in the negative direction . In two dimensions in the plane , we can move in infinitely many direction and precise means of describing direction is needed . The slope of the line segment joining $P=(x_1,y_1)$ to $Q=(x_2,y_2)$ show in figure below

$$m = [slope\ from\ P\ to\ Q\ ] = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$



If P and Q have the same x coordinate , then $x_1 = x_2$ and $\Delta x = 0$ .The line from P to Q vertical and the slope $m = \frac{\Delta y}{\Delta x}$ is undefined because $\Delta x=0$, if P and Q have the same y coordinate , then $y_1 = y_2$ and $\Delta y = 0$ .The line from P to Q horizontal and the slope $m = \frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$ assuming $\Delta x \neq 0$.

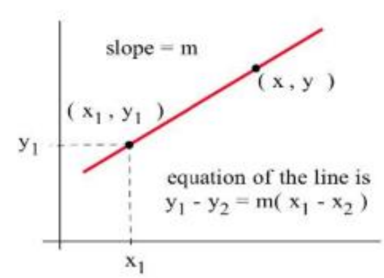
Equations of lines

Every line has the property that the slope of the segment between any two points on the line is the same , and this constant slope property of straight lines leads to ways of finding equations represent non-vertical lines.

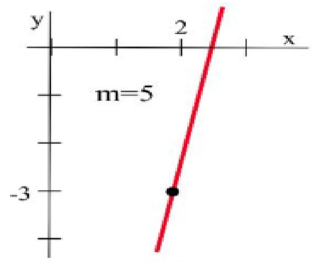
1- **Point – Slope equation**

If L is a non- vertical line through a known point $P=(x_1 , y_1)$ with known slope m show in figure below , then the equation of line L is

$$\boldsymbol{y_1 - y_2 = m\,(x_1 - x_2)}$$



Example: - Find the equation of the line through (2.5,-3) with slope 5



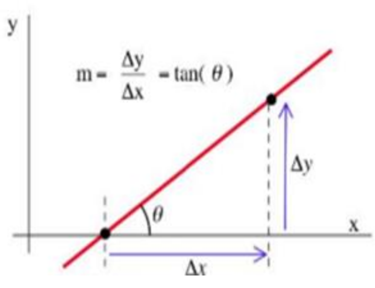
2- **Two –Point and slope – intercept equation**
If two points P=(x_1 , y_1) and Q=(x_2,y_2) are on the line L, then we can calculate the slope between them and use the first point and the point – slope equation of L:

$$Two\ Points : y - y_1 = m(x - x_1) where\ m = \frac{y_2 - y_1}{x_2 - x_1}$$

It is common practice to rewrite the equation of the line in the form y = mx + b , the slope – intercept from of the line . The line y =mx +b has slope m and crosses the y – axis at the point (0, b).

Angle between lines

The angle of inclination of a line with x – axis is the smallest angle θ which the line makes with the positive x – axis as measured from the x- axis counterclockwise to the line (Figure below) .Since the slope $\boldsymbol{m} = \frac{\Delta \boldsymbol{y}}{\Delta \boldsymbol{x}}$ and since $\mathbf{tan}(\boldsymbol{\theta}) = \frac{\boldsymbol{opposite}}{\boldsymbol{adjacent}}$, we have that $\boldsymbol{m} = \mathbf{tan}(\boldsymbol{\theta})$



The slope of the line is the tangent of the angle of inclination of the line.

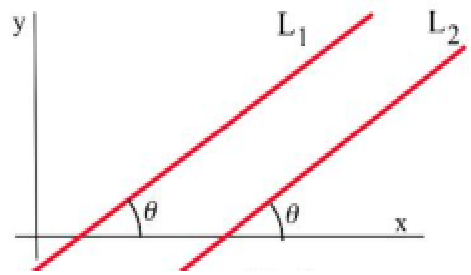
Parallel and Perpendicular Lines

Two Parallel lines L_1 and L_2 make equal angle with the x – axis so their angles of inclination will be equal (Figure below) and so will their slopes. Similarly. If their slopes m_1 and m_2 are equal, then the equations of the lines always differ by a constant:

$$y_1 - y_2 = (m_1x + b_1) = (m_2x + b_2)$$

$$= (m_1 - m_2)x + (b_1 - b_2)$$

$= b_1 - b_2$ Which is a constant so the lines will be parallel. These two ideas can be combined into a single statement.



Two non – vertical lines L_1 and L_2 with slopes m_1 and m_2 are parallel if and only if

$$m_1 = m_2$$

If lines L_1 and L_2 with slopes m_1 and m_2 are perpendicular if and only if

$$\boldsymbol{m_1 = -\frac{1}{m_2}}$$

Introduction to functions

A function is a rule which operates on one number to give another number. However, not every rule describes a valid function. This unit explains how to see whether a given rule describes a valid function, and introduces some of the mathematical terms associated with functions.

you should be able to:

- recognize when a rule describes a valid function,
- be able to plot the graph of a part of a function,
- find a suitable domain for a function, and find the corresponding range.

For example

$f(x) = x + 3.$

The number x that we use for the input of the function is called the argument of the function.

So if we choose an argument of 2, we get

$$f(2) = 2 + 3 = 5.$$

If we choose an argument of −6, we get

$$f(-6) = -6 + 3 = -3.$$

If we choose an argument of z, we get

$$f(z) = z + 3.$$

If we choose an argument of x^2, we get $f(x^2) = x^2 + 3.$

At first sight, it seems that we can pick any number we choose for the argument. However, that is not the case, as we shall see later. But because we do have some choice in the number we can pick, we call the argument the independent variable. The output of the function, e.g. f(x), f(5), etc. depends upon the argument, and so this is called the dependent variable.

> A function is a rule that maps a number to another unique number. The input to the function is called the **independent** variable, and is also called the argument of the function. The output of the function is called the **dependent** variable.

Domain and Range of a Function

The domain of a function is the complete set of possible values of the independent variable. **The domain is the set of all possible x-values which will make the function "work", and will output real y-values.**

<u>Note :</u>

- **The denominator (bottom) of a fraction cannot be zero**
- **The number under a square root sign must be positive in this section**

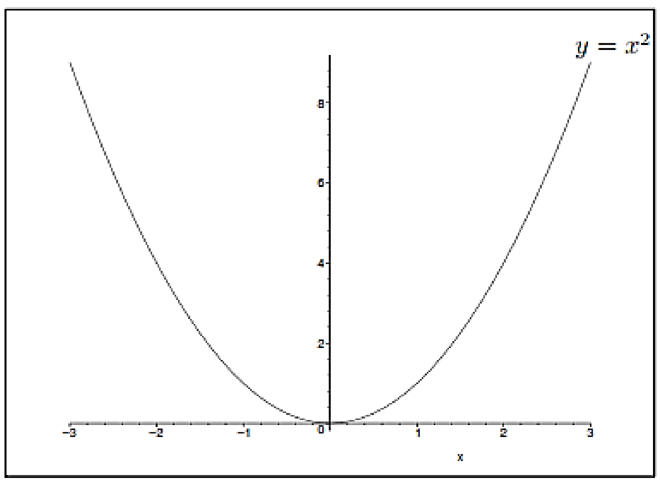
<u>The range</u> of a function is the complete set of all possible resulting values of the dependent variable (y, usually), after we have substituted the domain.

Even and Odd Functions

If a function f is such that $f(-x) = f(x)$, the function is said to be even. For example look at the function:

$$f(x) = x^2$$

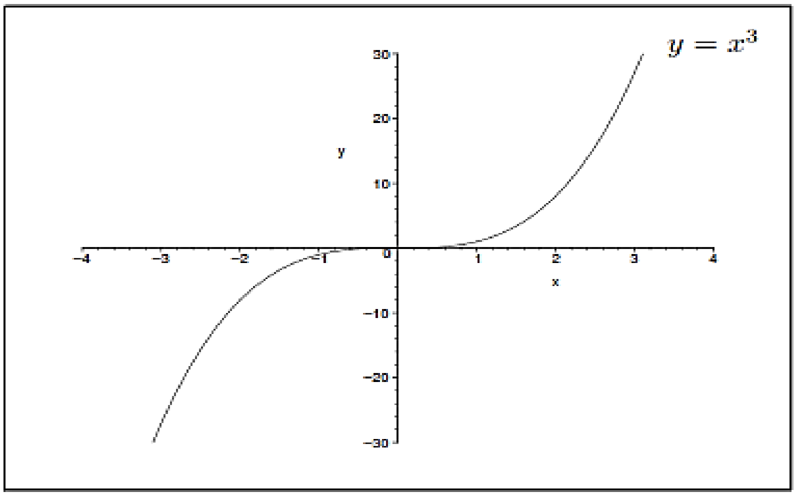
It is clear to see that$(-x)^2 = f(x)^2$, so the function that squares the input is an even function. The graph of an **<u>even function is symmetrical about the vertical coordinate axis (y-axis)</u>** as is displayed in the graph of the function $y = x^2$ below.



If a function f is such that $(-x) = -f(x)$, the function is said to be odd. For example look at the function :

$$f(x) = x^3$$

It is clear to see that $(-x)^3 = -(x)^3$, so the function that cubes the input is an odd function. The graph of an **<u>odd function is not symmetrical about the vertical coordinate</u>** axis (y-axis) as is displayed in the graph of the function $y = x^3$ below. **<u>It is however symmetrical through the origin.</u>**



Combining Functions

The topic with functions that we need to deal with is combining functions. For the most part this means performing basic arithmetic (addition, subtraction, multiplication, and division) with functions. There is one new way of combing functions that we'll need to look at as well.

Let's $f(x)$ and $g(x)$ functions with domains A and B .Then the functions we have the following notation and operations.

$(f+g)(x) = f(x) + g(x)$ $\quad$ $Domain\ A \cap B$

$(f-g)(x) = f(x) - g(x)$ $\quad$ $Domain\ A \cap B$

$(fg)(x) = f(x)g(x)$ $\quad$ $Domain\ A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\quad$ $Domain\ \{\, x \in A \cap B | g(x) \neq 0\}$

Sometimes we will drop the (x) part and just write the following,

$f + g = f(x) + g(x)$ $\quad$ $f - g\ = f(x) - g(x)$

$fg = f(x)g(x)$ $\quad$ $\frac{f}{g} = \frac{f(x)}{g(x)}$

Note as well that we put x in the parenthesis, but we will often put in numbers as well. Let's take a quick look at an example

Example: - Given $f(x) = 2 + 3x - x^2$ and $g(x) = 2x - 1$ evaluate each of the following

$(a)\quad (f + g)(4) \qquad (b)\ g - f \quad (c)(fg)(x) \qquad (d)\left[\frac{f}{g}\right](0)$

Solution

$(a)\quad (f + g)(4)$

In this case we've got a number so we need to do some function evaluation.

$$(f + g)(4) = f(4) + g(4)$$

$$= (2 + 3(4) - 4^2) + (2(4) - 1)$$

$$= -2 + 7$$

$$= 5$$

$(b)\ g - f$

Here we don't have an *x* or a number so this implies the same thing as if there were an *x* in parenthesis. Therefore, we'll subtract the two functions and simplify. Note as well that this is written in the opposite order from the definitions above, but it works the same way.

$$g - f = g(x) - f(x)$$

$$= 2x - 1 - (2 + 3x - x^2)$$

$$= 2x - 1 - 2 - 3x + x^2$$

$$= x^2 - x - 3$$

$(c)(fg)(x)$

As with the last part this has an *x* in the parenthesis so we'll multiply and then simplify.

$$(fg)(x) = f(x)g(x)$$

$$= (2 + 3x - x^2)(2x - 1)$$

$$= 4x + 6x^2 - 2x^3 - 2 - 3x + x^2$$

$$= -2x^3 + 7x^2 + x - 2$$

$(d)\left[\frac{f}{g}\right](0)$

In this final part we've got a number so we'll once again be doing function evaluation.

$$\left[\frac{f}{g}\right](0)$$

$$= \frac{f(0)}{g(0)}$$

$$= \frac{2 + 3(0) - (0)^2}{2(0) - 1} = \frac{2}{-1}$$

Example: - The domain of $f(x) = \sqrt{x}\ is\ A = [0, \infty), the\ domain\ of\ g(x) = \sqrt{1 - x}\ is\ B = (-\infty, 1], and\ the\ domain\ of\ h(x) = \sqrt{x - 1}$ is C =[1,∞) , so the domain of

$$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}\ is\ A \cap B = [0,1]$$

and

$$(f - h)(x) = \sqrt{x} - \sqrt{x - 1}\ is\ A \cap C = [1, \infty)$$

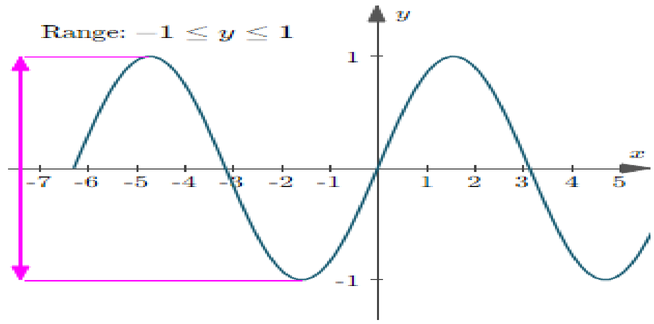
Example: - The domain of $f(x) = x^2\ and\ g(x) = x - 1$ the domain of the rational function $\left(\frac{f}{g}\right)(x) = \frac{x^2}{x-1}$ is $\{ x \in |x \neq 1\}\ or\ (-\infty, 1) \cup (1, \infty)$.

Example:- $\boldsymbol{y = \sqrt{x + 4}}$

1- The **domain** of this function is x ≥ -4, since x cannot be less than -{4}; this will make the number under the square root negative.
2- The **range** in this case is $y \geq 0$.

Example: - From graph of the curve y = sin x Find

1- Domain of function
2- Range of function



Answer:

1- The **domain** of $y =$ sin x is "all values of x", since there are no restrictions on the values for x. y
2- From observing the curve, we can see the range is y between – 1 and 1. We could write this as $-1 \le y \le 1$.

Plotting the graph of a function

If we have a function given by a formula, we can try to plot its graph. Suppose, for example, that we have a function f defined by

$$f(x) = 3x^2 - 4.$$

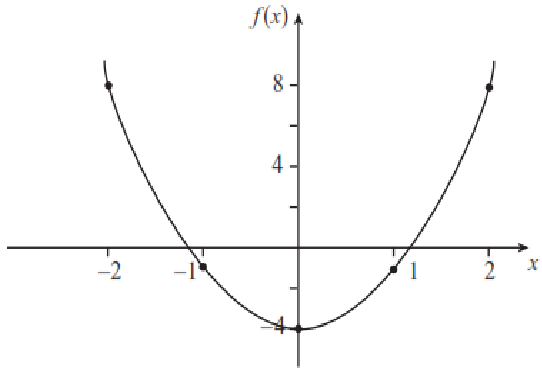
The argument of the function (the independent variable) is x, and the output (the dependent variable) is $3x^2 - 4$.

Solution :-

1- The domain all values of x
2- The range $-\infty \le y \le \infty$

So we can calculate the output of the function for different arguments:

Function	Value of (x) Domain	Value of (y) Range
$F(0)= 3 \times (0)^2 - 4$	0	– 4
$F(1)= 3 \times (1)^2 - 4$	1	– 1
$F(2)= 3 \times (2)^2 - 4$	2	8
$F(-1)= 3 \times (-1)^2 - 4$	–1	–1
$F(0)= 3 \times (-2)^2 - 4$	–2	8



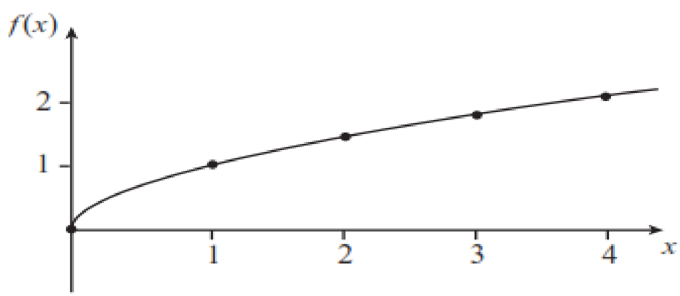
Example: - Plot the function $f(x) = \sqrt{x}$

Solution: -

1- The domain x ≥ 0, f(x) ≥ 0.

2- The range the range is the corresponding set of numbers y ≥ 0.

Function	**Value of (x) Domain**	**Value of (y) Range**
F(0)= $\sqrt{0}$	0	0
F(1)= $\sqrt{1}$	1	1
F(2)= $\sqrt{2}$	2	1.4142
F(3)= $\sqrt{3}$	3	1.732
F(4)=$\sqrt{4}$	4	2



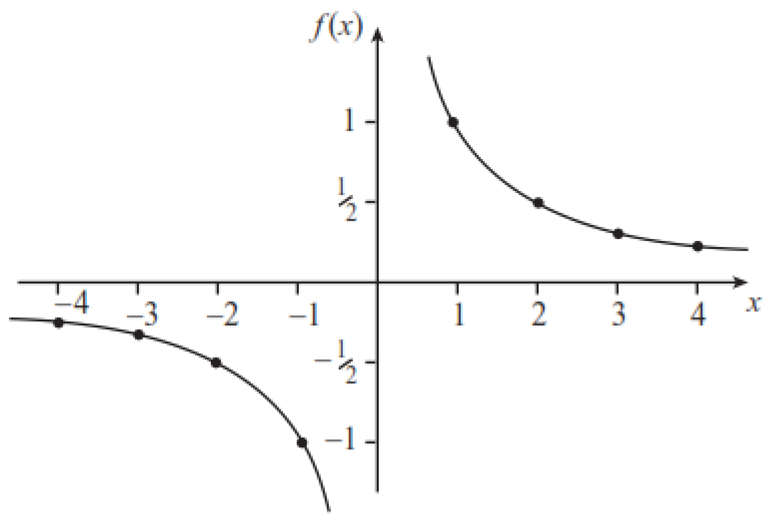
Example: - Plot the function

$$f(x) = \frac{1}{x}$$

Solution:-

1- The domain the domain all values of x / {0} , (- ∞ , 0) ∪ (0, ∞) .
2- The range (- ∞ , 0) ∪ (0, ∞)

Function	**Value of (x) Domain**	**Value of (y) Range**
F(1)= $\frac{1}{1}$	1	1
F(2)= $\frac{1}{2}$	2	0.5
F(3)= $\frac{1}{3}$	3	0.333
F(4)= $\frac{1}{4}$	4	0.25
F(-1)=$\frac{1}{-1}$	-1	-1
F(-2)=$\frac{1}{-1}$	-1	-1
F(-2)=$\frac{1}{-2}$	-2	- 0.5
F(-3)=$\frac{1}{-1}$	-3	- 0.333
F(-1)=$\frac{1}{-1}$	-4	- 0.25



Exercises

1- Consider the function

$$f(x) = \frac{1}{(x-2)^2}$$

(a) Write down the argument of this function.
(b) Write down the dependent variable in terms of the argument.
(c) Use a table of values to help you plot the graph of the function.

2 - Consider the function

$$f(x) = 2x^2 + 5x - 3$$

(a) Write down the argument of this function.
(b) Write down the dependent variable in terms of the argument.
(c) Use a table of values to help you plot the graph of the function.
(d) From your graph, estimate f(1.5).
(e) Use your function to calculate f(1.5) exactly.
(f) Write down the domain and range of the function.
(g) Re-write the function with argument y.

3 - Consider the function

$$f(x) = \frac{1}{(x-3)^2}$$

(a) Plot the graph of the function.
(b) Write down the domain and range of the function.
(c) Re-write the function with argument z.
(d) Use your graph to estimate f(1).
(e) Use the function to calculate f(1) exactly.
(f) Write down another function where x = 4 has to be omitted from the domain.

4. **Consider the function**

$$f(x) = 3\sqrt{x}$$

(a) Write down the argument of this function.
(b) Write down the dependent variable in terms of the argument.
(c) Use a table of values to help you plot the graph of the function.

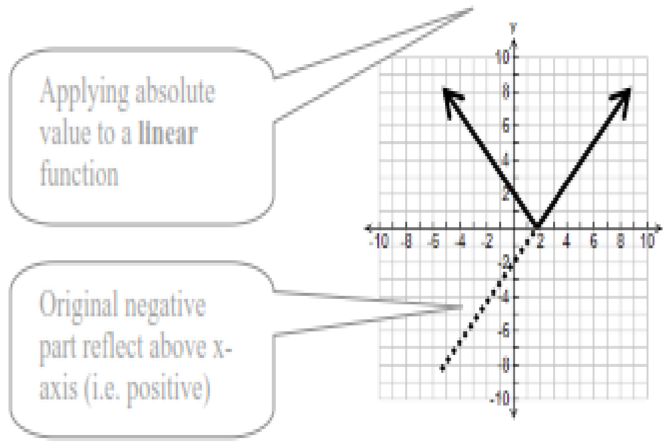
Absolute Value Inequalities

Graphically the **absolute value function** reflects any part below the x-axis to the positive side.

Example

a) graph $f(x) = |x - 2|$

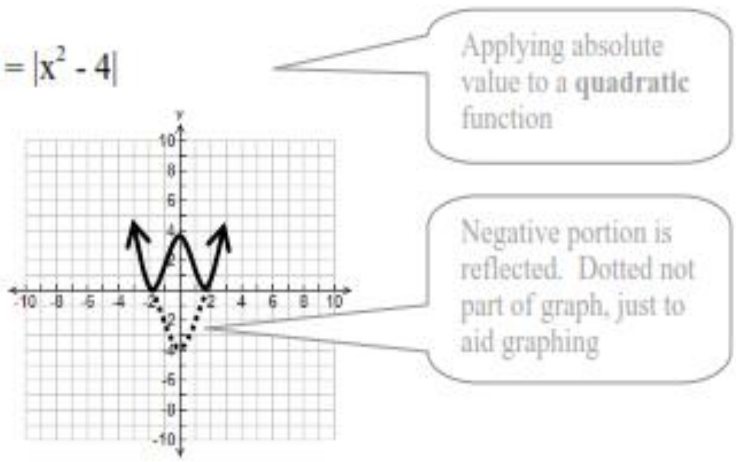
$x-2 \geq 0 \rightarrow x \geq 2$	$F(x)=x-2$	(x,y)
2	0	(2,0)
3	1	(3,1)
4	2	(4,2)
5	3	(5,3)
$x-2<0 \rightarrow x < 2$	$F(x) =-(x-2)$	(x,y)
-3	5	(-3,5)
-4	6	(-4,6)
-5	7	(-5,7)



b) $g(x) = |x^2 - 4|$

$x^2-4 \geq 0 \rightarrow x^2 \geq 4$ $x \geq \pm 2 \rightarrow x \geq 2 , x \leq -2$	$F(x) = x^2 - 4$	(x,y)
2	0	(2 , 0)
3	5	(3, 5)
4	12	(4,12)
-2	0	(-2,0)
-3	5	(-3,5)
-4	12	(-4,12)
$x^2-4<0 \rightarrow x^2 < 4$ $x < \pm 2 \rightarrow x<2 , x > -2$	$F(x) =-(x^2-4)$	(x,y)
1	3	(1,3)
0	4	(0,4)
-1	3	(-1,3)

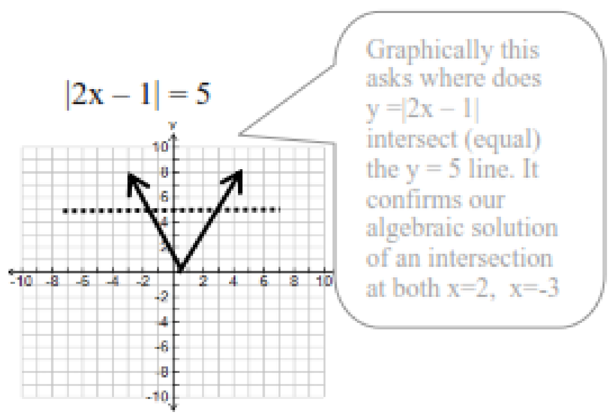
b) $g(x) = |x^2 - 4|$



c) $|2x – 1| = 5$

case 1: Negative	**case 2: Positive**
-(2x – 1) = 5	**(2x – 1) = 5**
-2x + 1 = 5	**2x – 1 = 5**
x = 2	**x=3**

2x -1 ≥ 0 $x \geq \frac{1}{2}$	**y= 2x – 1**	**(x, y)**
1/2	0	(1/2 ,0)
1	1	(1,1)
2	3	(2,3)
2x -1 <0 $x < \frac{1}{2}$	**y= 1-2x**	**(x, y)**
0	1	(0,1)
-1	3	(-1,3)
-2	5	(-2,5)
-3	7	(-3,7)



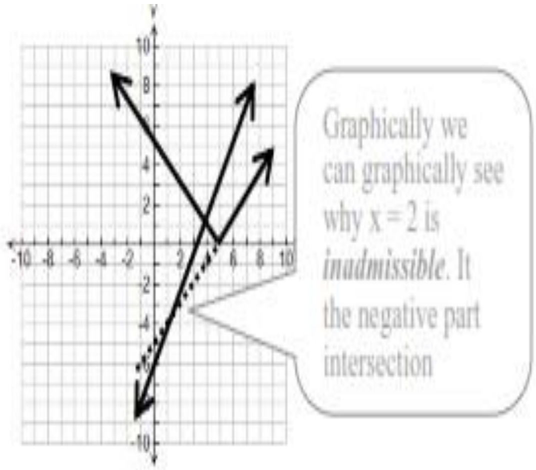
d) $|x – 5|=2x – 7$

The algebra seem to work fine, but one needs to be careful. x = 2 is ***inadmissible***. The algebra seem to work fine, but one needs to be careful. x = 2 is ***Inadmissible***.

-+*

case 1: Positive	**case 2: Negative**
(x – 5) = 2x – 7	**-(x – 5) = 2x – 7**
x = 2	**3x=12**
	x=4
	∴ x = 4 is only solution

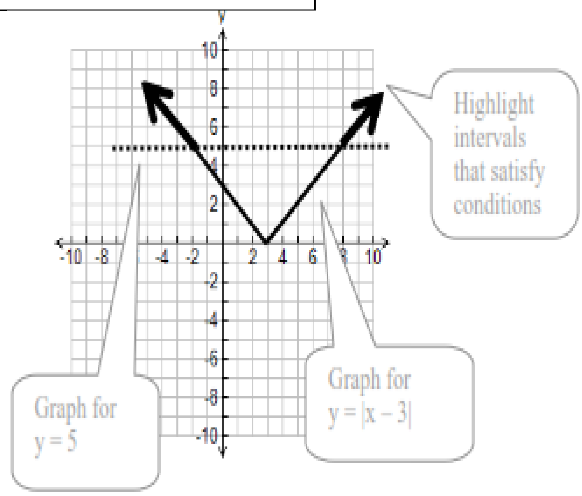
$x - 5 \geq 0$ $x \geq 5$	$y = x - 5$	**(x, y)**
5	0	(5,0)
6	1	(6, 1)
7	2	(7, 2)
$x - 5 < 0$ $x < 5$	**$y = -x + 5$**	**(x, y)**
4	1	(4 ,1)
3	2	(3, 2)
2	3	(2,3)
1	4	(1,4)
0	5	(0,5)
-1	6	(-1,6)
-2	7	(-2 ,7)



e) $|x - 3| > 5$

case 1: Negative **$-(x - 3) > 5$** **$-x > 2$** **$x < -2$**	**case 2: Positive** **$(x - 3) > 5$** **$x > 8$**
both solutions confirmed graphically $\therefore x < -2$ or $x > 8$	

$x - 3 \geq 0$ $x \geq 3$	$y = x - 3$	**(x, y)**
3	0	(3,0)
4	1	(4,1)
5	2	(5 ,2)
6	3	(6,3)
$x - 3 < 0$ $x < 3$	**$y = -x + 3$**	**(x, y)**
2	1	(2,1)
1	2	(1,2)
0	3	(0,3)
-1	4	(-1 ,4)
-2	5	(-2,5)



1. Graph the following.

a) $f(x) = |x + 2|$ b) $f(x) = |x - 2|$ c) $f(x) = |3x - 1|$

d) $h(x) = |x| + 2$ e) $h(x) = 2|x| - 1$ f) $h(x) = \frac{1}{2}|x + 1| - 3$

g) $g(x) = |x^2 - 3|$ h) $g(x) = |(x - 3)^2|$ i) $g(x) = |(x + 2)(x - 1)|$

j) $y = |(x + 3)^2 (x - 2)|$ k) $y = |-2x^2 + 4x|$

l) $y = -|(x + 1) - 2| + 3$

2. Solve the following equations algebraically and check graphically

a) $|x + 4| = -2$ b) $|x - 4| = 3x - 7$

c) $3x - |x + 3| = 0$ d) $4x + 1 = |x - 5|$

3. Solve the following inequalities.

a) $|x| < 3$ b) $|x - 2| \leq 3$ c) $|x + 3| \geq -1$

d) $5 \leq |x - 4|$ e) $|x - 4| > 3x - 7$ f) $2x - |x + 3| < 0$.

Linear functions

A linear function is a function of the form $f(x) = ax + b$, where a and b are real numbers. Here, a represents the gradient of the line (slope), and b represents the y-axis intercept (which is sometimes called the vertical intercept).So we only need two points to be able to draw the line. However, we generally choose three, and the third point is a good check that we haven't made a mistake.

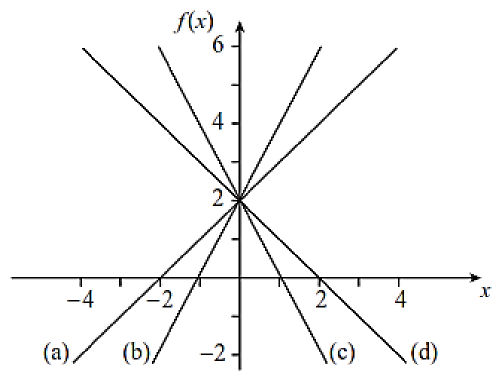
1- Example

(a) $f(x) = x + 2 : f(0) = 2, f(1) = 3, f(2) = 4$

(b) $f(x) = 2x + 2 : f(0) = 2, f(1) = 4, f(2) = 6$

(c) $f(x) = -2x + 2 : f(0) = 2, f(1) = 0, f(2) = -2$

(d) $f(x) = -x + 2 : f(0) = 2, f(1) = 1, f(2) = 0$

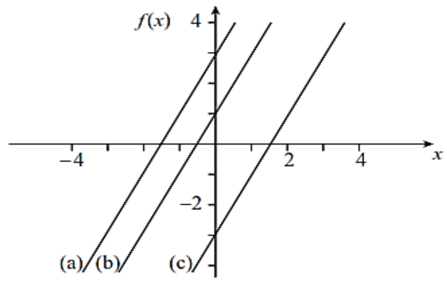


2- Example

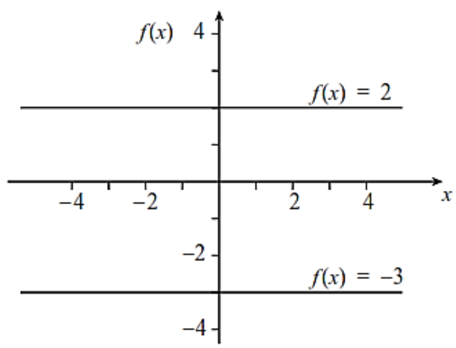
$(a)\quad f(x) = 2x + 3 : f(0) = 3, f(1) = 5, f(2) = 7$

$(b)\quad f(x) = 2x + 1 : f(0) = 1, f(1) = 3, f(2) = 5$

$(c)\quad f(x) = 2x - 3 : f(0) = -3, f(1) = -1, f(2) = 1$



Then we would have functions of the form $f(x) = b$ where b is constant, for example $f(x) = 2$ or $f(x) = -3$.



Note

1- You can see that a **<u>gradient of zero always gives a horizontal line</u>**, and that the line cuts the y-axis at b.

2- The y-axis intercept would be equal to zero, and so the graphs of all these functions pass through the origin, and the **<u>gradient of the line depends upon a.</u>**

3- Parallel lines have **<u>the same gradient</u>**

Definition

Functions of the form

$$f(x) = ax + b$$

are linear, and they are represented graphically by straight lines. The number a represents the gradient of the line, and the number b represents the y-axis intercept.

<u>Exercises</u>

1. What is a linear function?

2. By drawing up a table of values, plot the following linear functions on the same axes:

$(a)\ f(x) = 2x + 1$ $(b)\ f(x) = 3x - 2$ $(c)\ f(x) = 4 - 3x$ $(d)\ f(x) = 2 - x$

3. Find the gradient and the vertical intercept for each of the following linear functions by re- arranging them into the form f(x) = ax + b

<u>(note: y = f(x)).</u>

$(a)\ 2y + 4x = 12$ $(b)\ 5x - y = 9$ $(c)\ -3x = 1 - 4y$ $(d)\ 2 - y/3 = x$

$(e)\ 3 = 3y/4 - 2x/3$ $(f)\ 12x - 4 = y/3 + 3$

4. Write down three different functions in which all the graphs are represented by parallel lines.

5. Write down three different functions in which all the graphs have the same vertical intercept.

Trigonometric Functions

A right triangle is a triangle with a right angle (90°) (See Figure below).

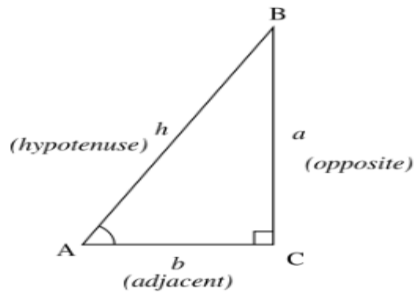


Figure: - Right triangle

For angle θ, the trigonometric functions are defined as follows:

$sin\theta = \frac{opposite}{hypotenuse}$	$cos\theta = \frac{adjacent}{hypotenuse}$	$tan\theta = \frac{opposite}{adjacent}$
$csc\theta = \frac{hypotenuse}{opposite}$ or $csc\theta = \frac{1}{sin\theta}$	$sec\theta = \frac{hypotenuse}{adjacent}$ or $sec\theta = \frac{1}{cos\theta}$	$cot\theta = \frac{adjacent}{opposite}$ or $cot\theta = \frac{1}{tan\theta}$

Relationship between Degrees and Radians

A radian is defined as an angle subtended at the center of a circle for which the arc length is θ equal to the radius of that circle (see figure)

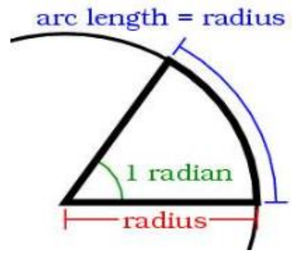


Figure Definition of a radian.

The circumference of the circle is equal to 2πr , where r is the radius of the circle. Consequently, $360^\circ = 2\,\pi\, radians$.Thus convert from radian to degree

$$\mathbf{1\, radian = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}}$$

Where convert from degree to radian

$$\mathbf{1\, degree = \frac{\pi}{180^\circ}}$$

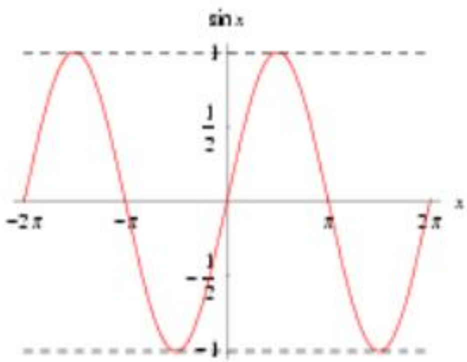
Inverse Trigonometric Functions

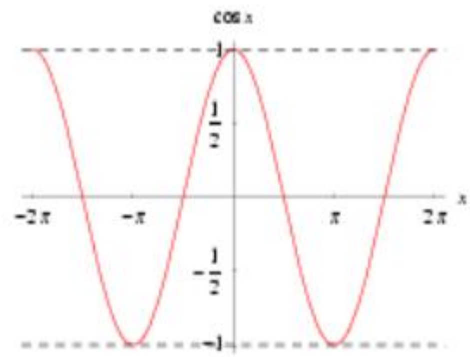
If $x = sin(\theta), Then\ \theta = sin^{-1}(x)$, i.e. s is the angle whose sine is x. In other words, θ is the inverse sine of x. Another name for inverse sine is arcsine, and the notation used is **θ=arcsin(x)**. Similarly, we can define inverse cosine, inverse tangent, inverse cotangent, inverse secant and inverse cosecant.

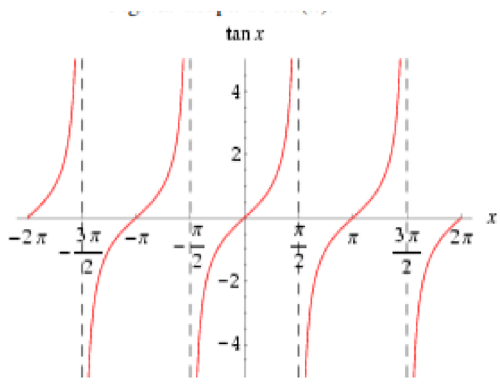
$x = sin\theta$	$\theta = sin^{-1}(x) = \text{arcsin}(x)$
$x = cos\theta$	$\theta = cos^{-1}(x) = \text{arccos}(x)$
$x = tan\theta$	$\theta = tan^{-1}(x) = \text{arctan}(x)$
$x = csc\theta$	$\theta = csc^{-1}(x) = \text{arccsc}(x)$
$x = csc\theta$	$\theta = csc^{-1}(x) = \text{arccsc}(x)$
$x = sec\theta$	$\theta = sec^{-1}(x) = \text{arcsec}(x)$
$x = cot\theta$	$\theta = cot^{-1}(x) = \text{arccot}(x)$

Graphs of Trigonometric Functions

Note: In each graph in Figure below, the horizontal axis (x) is **measured in radians**. "Sine." "Cosine." "Tangent."

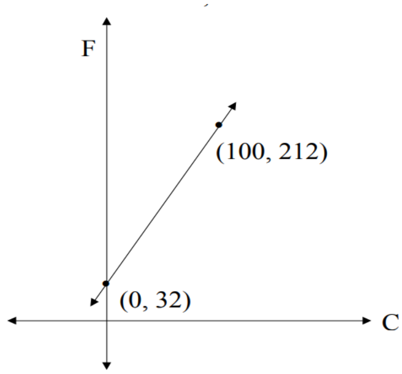






Lines in the Plane

The formula y = mx + b sometimes appears with different symbols. They described a variety of physical, biological and business phenomena such $d = rt$ relating the distance d traveled to the rate r and time t of travel, and $C = \frac{5}{9}\ (F - 32)\ or\ F = \frac{9}{5}c + 32$ for converting the temperature in Fahrenheit degree (F) to Celsius (C) .Basic Temperature Facts Water freezes at: 0°C, 32°F Water Boils at: 100°C, 212°F



To find the equation relating Fahrenheit to Celsius we need m and b

$$m = \frac{F2 - F1}{C2 - C1} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}$$

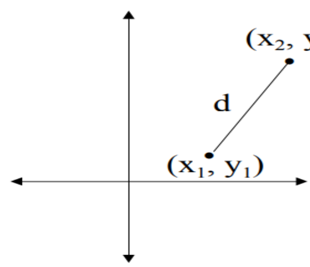
Therefor $F = \frac{9}{5} C + b$ To find b, substitute the coordinates of either point.

$$32 = \frac{9}{5} \times 0 + b \rightarrow b = 32 \ Therefore \ \boldsymbol{F = \frac{9}{5} C + 32}$$

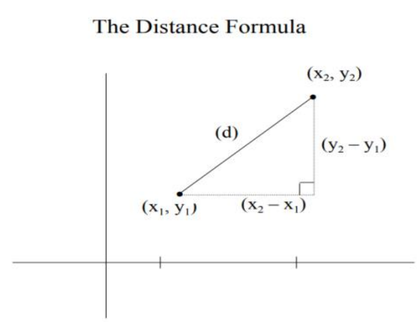
The Distance Formula

The distance between two points (x_1 , y_1) and (x_2 , y_2) is given by the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



This comes from the Pythagorean Theorem.

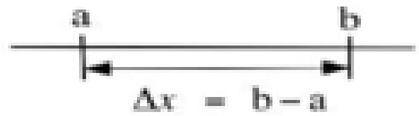


From Pythagoras' Theorem

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If we move from a point x=a to point b on line (Figure 2) , then we will have moved an increment of b – a .This increment is denoted by the symbol Δx (read " delta x ") . If b is larger than a , then we will have moved in the positive direction , and $\Delta x = b - a$ will be positive . If b is smaller than a , then we will be negative and we will have moved in the negative direction , Finally , if $\Delta x = b - a$ is zero , then a = b and did not move at all



We can also use the Δ notation and absolute values to write the distance that we have moved . on the number line , the distance from x = a to x =b is

dist(a, b) = $\begin{cases} \boldsymbol{b - a \text{ if } b \geq a} \\ \boldsymbol{a - b \text{ if } b < a} \end{cases}$ **or simply , dist(a,b) =** $|\boldsymbol{b - a}| = |\Delta \boldsymbol{x}| = \sqrt{(\Delta \boldsymbol{x})^2}$

The midpoint of the segment from x = a to x = b is the point $\boldsymbol{M = \frac{a+b}{2}}$ **on the number line**

Example :- find the length and midpoint of the interval x = - 3 to x= 6

Solution :- dist(-3,6) = $|\boldsymbol{6 - (-3)}| = |\boldsymbol{9}|$ **= 9 . the point** $\boldsymbol{M = \frac{-3+6}{2} = \frac{3}{2}}$

Example :- find the length and midpoint of the interval x = - 7 to x= -2 .

Exponential Function(y= a^x)

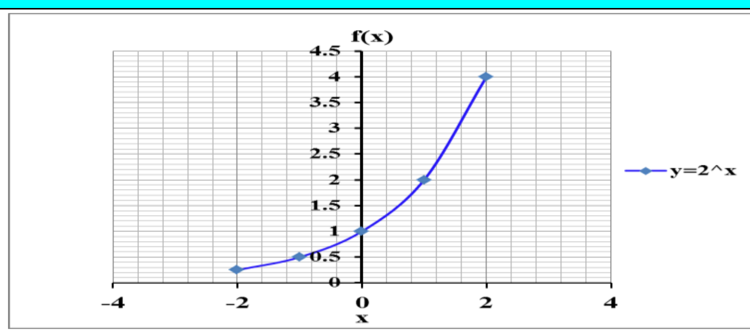
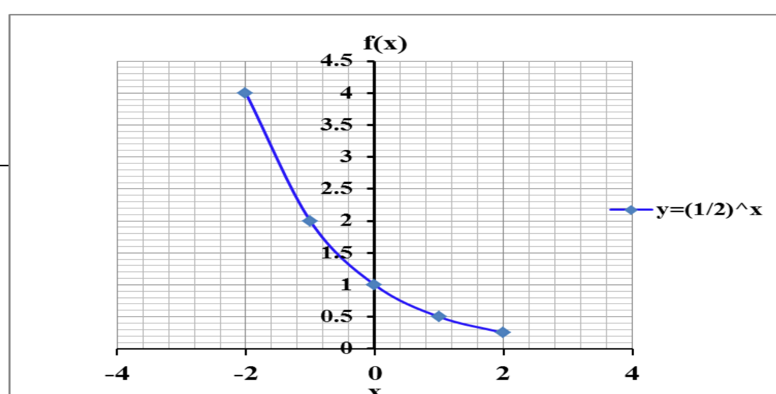
Exponential functions are one of the most important functions in mathematics. Exponential functions have many scientific applications, such as population growth and radioactive decay. Exponential functions are function where the variable x is in the exponent. Some examples of exponential functions are $f(x) = 2^x$, $f(x) = 5^{x-2}$, or $f(x) = 9^{2x+1}$. In each

An **Exponential Function** is a function of the

form $f(x) = b^x$ or $y = b^x$

where b is called the “base” and b is a positive real number other than ($b > 0$ and $b \neq 1$). The domain of an exponential function is all real numbers, that is, x can be any real number. **Note Domain (-∞ ,** examples the variable x is in the exponent, which makes each of the examples exponential functions.

Graphing Exponential Functions

<table>
<tr><th colspan="3">Example 1: Graph $f(x) = 2^x$</th></tr>
<tr><td>x</td><td>$f(x)= 2^x$</td><td rowspan="6">
</td></tr>
<tr><td>-2</td><td>$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$</td></tr>
<tr><td>-1</td><td>$f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$</td></tr>
<tr><td>0</td><td>$f(0) = 2^0 = 1$</td></tr>
<tr><td>1</td><td>$f(1) = 2^1 = 2$</td></tr>
<tr><td>2</td><td>$f(2) = 2^2 = 4$</td></tr>
<tr><th colspan="3">Example 2: Graph $f(x) =(\frac{1}{2})^x$</th></tr>
<tr><td>x</td><td>$f(x) =(\frac{1}{2})^x$</td><td rowspan="2">
</td></tr>
<tr><td>-2</td><td>$f(-2) = (\frac{1}{2})^{-2}=(\frac{1^{-2}}{2^{-2}})$ =4</td></tr>
</table>

-1	**f(-1) = $(\frac{1}{2})^{-1}$=$(\frac{1^{-1}}{2^{-1}})$ =2**	
0	**f(0) = $(\frac{1}{2})^{0}$=1**	
1	**f(1) = $(\frac{1}{2})^{1}$=$(\frac{1^{1}}{2^{1}})$ =$\frac{1}{2}$**	
2	**f(2) = $(\frac{1}{2})^{2}$=$(\frac{1^{2}}{2^{2}})$ =$\frac{1}{4}$**	

Properties of Exponential Functions

1.a^m a^n = a^{m+n}	**2.$(a^m)^n$ = a^{mn}**	**3. $(ab)^m$ = a^m b^m**
4.$\frac{a^m}{a^n}$ = a^{m-n}, a $\neq 0$	**5.$(\frac{a}{b})^m$ =$\frac{a^m}{b^m}$, b $\neq$0**	**6. a^{-m} = $\frac{1}{a^m}$, a$\neq$ 0**
7. $a^{1/n}$ = $\sqrt[n]{a}$	**8. a^0= 1 , a $\neq$ 0**	**9. $a^{m/n}$ = $\sqrt[n]{a^m}$ = $(\sqrt[n]{a})^m$**

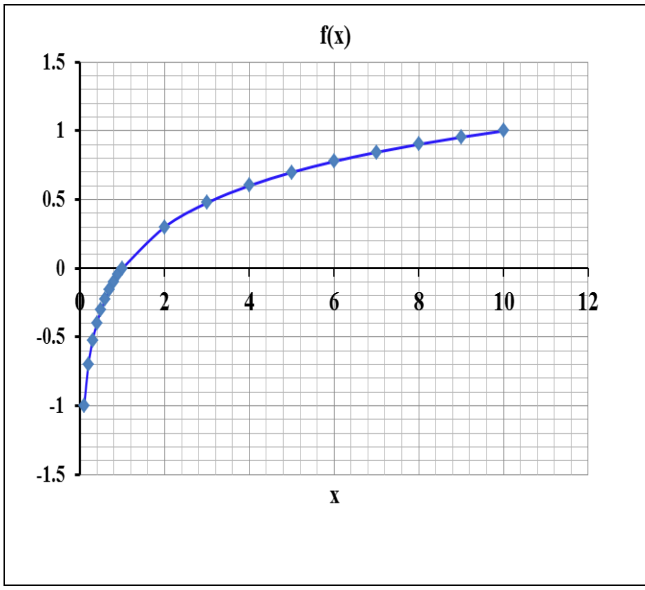
Example:-

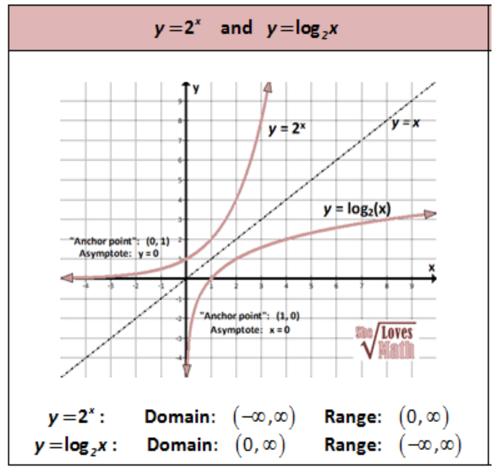
1- **Graph y = 3^{-x} -1**
2- **Graph y= $\left[\frac{2}{3}\right]^{x+1} - 2$**

Logarithmic Functions ($\log_a x$)

In the previous section we observed that exponential functions are one-to-one; this implies that they have inverse functions under composition. **Definition** the logarithmic function g with base a is the inverse of the function f(x) = a^x for a > 0, a ≠ 1. We write **g(x) = $\log_a$(x)** That is **y = $\log_a$(x)** if and only if **a^y=x** .The domain of $\log_a$ is is (0, ∞), and the range is (−∞, ∞).

y = $\log_a$(x) , where (0 < x < ∞)	**y= a^x , y= $\log_a$ x**





Exponential Functions ⟺ **Logarithmic Functions**

$y = b^x$ ⟺ $x = \log_b y$

$(b > 0, b \neq 1, y > 0)$

Example: $25 = 5^2$

Example: $2 = \log_5 25$

Properties of Logarithm Function

An important point to note here is that, regardless of the argument $2^{f(x)} > 0$.

So we shall consider only positive arguments.

1. $log_b(mn) = log_b(m) + log_b(n)$
2. $log_b(^m/_n) = log_b(m) - log_b(n)$
3. $log_b(m^n) = n \cdot log_b(m)$
4. $\log_a 1 = 0$ because $a^0 = 1$
   No matter what the base is, as long as it is legal, the log of 1 is always 0. That's because logarithmic curves always pass through (1,0)
5. $\log_a a = 1$ because $a^1 = a$
   Any value raised to the first power is that same value.
6. $\log_a a^x = x$
   The log base a of x and a to the x power are inverse functions. Whenever inverse functions are applied to each other, they inverse out, and you're left with the argument, in this case, x.

7. $\log_a x = \log_a y$ implies that $x = y$
   If two logs with the same base are equal, then the arguments must be equal.
8. $\log_a x = \log_b x$ implies that $a = b$

If two logarithms with the same argument are equal, then the bases must be equal.

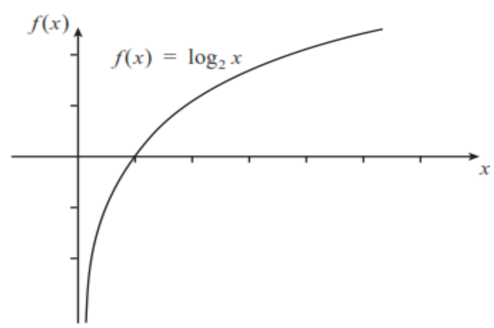
Suppose that a = 2. Then $\boldsymbol{f(x) = log_2 x \quad mean \ (2)^{f(x)} = x}$
An important point to note here is that, regardless of the argument, $(\frac{1}{2})^{f(x)} > 0$. So we shall consider only positive arguments.

$$f(x) = (2)^x$$

$f(1) = log_2 1$	mean $2^{f(1)} = 1$	so $f(1) = 0$
$f(2) = log_2 2$	mean $2^{f(2)} = 2$	so $f(2) = 1$
$f(4) = log_2 4$	mean $2^{f(4)} = 2^2$	so $f(4) = 2$
$f(\frac{1}{2}) = log_2(\frac{1}{2})$	mean $2^{f(\frac{1}{2})} = \frac{1}{2} = 2^{-1}$	so $f(\frac{1}{2}) = -1$
$f(\frac{1}{4}) = log_2(\frac{1}{4})$	mean $2^{f(\frac{1}{4})} = (\frac{1}{4}) = 2^{-2}$	so $f(\frac{1}{4}) = -2$

We can put these results into a table, and plot a graph of the function.

x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2



Suppose that a $=\frac{1}{2}$. Then $\boldsymbol{f(x) = log_{1/2} x \quad mean \ (\frac{1}{2})^{f(x)} = x}$
An important point to note here is that, regardless of the argument, $(\frac{1}{2})^{f(x)} > 0$. So we shall consider only positive arguments.

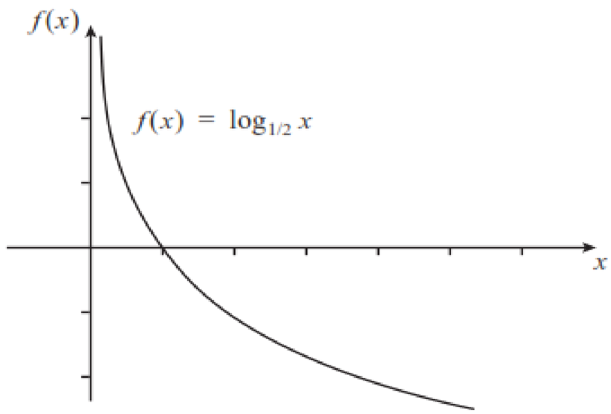
$$f(x) = (\frac{1}{2})^x$$

$f(1) = log_{1/2} 1$	mean $(\frac{1}{2})^{f(1)} = 1$	so $f(1) = 0$

$f(2) = \log_{1/2} 2$	mean $(\frac{1}{2})^{f(2)} = (\frac{1}{2})^{-1}$	so $f(2) = -1$
$f(4) = \log_{1/2} 4$	mean $(\frac{1}{2})^{f(4)} = (\frac{1}{2})^{-2}$	so $f(4) = -2$
$f(\frac{1}{2}) = \log_{1/2}(\frac{1}{2})$	mean $2^{f(\frac{1}{2})} = \frac{1}{2}$	so $f(\frac{1}{2}) = 1$
$f(\frac{1}{4}) = \log_{1/2}(\frac{1}{4})$	mean $2^{f(\frac{1}{4})} = (\frac{1}{2})^2$	so $f(\frac{1}{4}) = 2$

We can put these results into a table, and plot a graph of the function.

x	$f(x)$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	−1
4	−2



- **Expand the following:**

$$\log_3\left(\frac{4(x-5)^2}{x^4(x-1)^3}\right)$$

Use the log rules:

$$\begin{aligned}\log_3\left(\frac{4(x-5)^2}{x^4(x-1)^3}\right) &= \log_3\left(4(x-5)^2\right) - \log_3\left(x^4(x-1)^3\right)\\ &= \left[\log_3(4) + \log_3\left((x-5)^2\right)\right] - \left[\log_3\left(x^4\right) + \log_3\left((x-1)^3\right)\right]\\ &= \log_3(4) + \log_3\left((x-5)^2\right) - \log_3\left(x^4\right) - \log_3\left((x-1)^3\right)\\ &= \log_3(4) + 2\log_3(x-5) - 4\log_3(x) - 3\log_3(x-1)\end{aligned}$$

Change of base

The logarithm $\log_b(x)$ can be computed from the logarithms of x and b with respect to an arbitrary base k using the following formula:

$$\log_b(x)=\frac{\log_d(x)}{\log_d(b)}$$

- **Evaluate $log_3(6)$.**

$$\log_3(6)=\frac{\log(6)}{\log(3)}=\frac{0.778151250384\ldots}{0.47712125472\ldots}=1.63092975357\ldots$$

Natural logarithm (ln) & inverse function of ln (e^x)

Definition of natural logarithm

The **natural logarithm** of a number is its logarithm to the base of the mathematical constant *e*, where *e* is number approximately equal to $e \approx 2.718281828459$.

The natural logarithm of *x* is generally written as ln *x* , $\log_e x$.

$$x = e^y \rightarrow y = \log_e x = \ln x = \text{Natural Logarithmic}$$

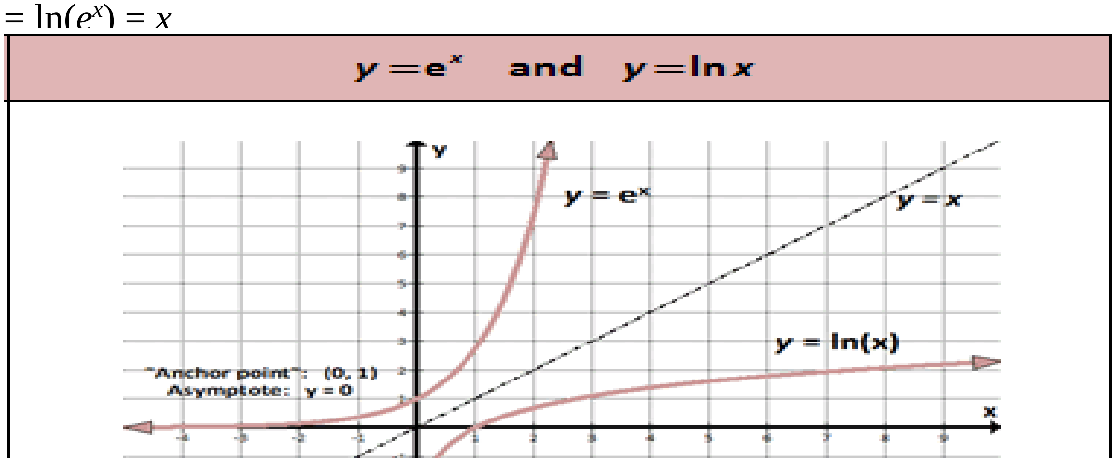
- $\ln x = \log_e x = \frac{\log_{10} x}{\log_{10} e}$

In as inverse function of exponential function

The natural logarithm function ln(x) is the inverse function of the exponential function e^x.

For x>0,

$f(f^{-1}(x)) = e^{\ln(x)} = x$ or $f^{-1}(f(x)) = \ln(e^x) = x$



Natural logarithm rules and properties

Rule name	Rule lnx	Rule e^x
Product	$\ln(x \cdot y) = \ln(x) + \ln(y)$	$e^x . e^y = e^{x+y}$
Quotient	$\ln(x / y) = \ln(x) - \ln(y)$	$e^x/e^y = e^x - e^y$
Power rule	$\ln(x^y) = y \cdot \ln(x)$	$(e^x)^y = e^{xy}$
	$e^{\ln(x)} = x$	$\ln(e^x) = x$
ln of negative number	$\ln(x)$ is undefined when $x \leq 0$	$e^{-a} = \frac{1}{e^a}$
ln of zero	$\ln(0)$ is undefined	$e^0 = 1$
ln of one	$\ln(1) = 0$	

Example :-

1- $ln\frac{e^{2x}}{5} = \ln e^{2x} - \ln 5 = 2x - \ln 5$

2- Solve for y

a- $\ln y = x^2$

Solution

$e \ln y = e^{x^2}$

$y = e^{x^2}$

b- $e^{3y} = (2 + \cos x)$

Solution

$\ln e^{3y} = \ln(2 + \cos x)$

$3y = \ln(2 + \cos x)$

$$y = \frac{\ln(\,2+cosx\,)}{3}$$

- **Convert *ln*(4) to an expression written in terms of the common log.**

$$ln(4) = \frac{log(4)}{log(e)}$$

H.W:-

$1 - \ e^{\sqrt{y}} = \ x^2$

2 – ln(y – 2) = ln (sinx) – x

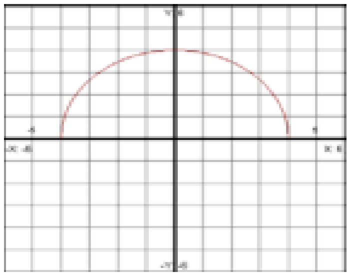
One-Sided Limits

$\lim\limits_{x \to c^+} f(x) = L$ This means we are finding the limit of f as we approach c from the right (positive side).

$\lim\limits_{x \to c^-} f(x) = L$ This means we are finding the limit of f as we approach c from the left (negative side).

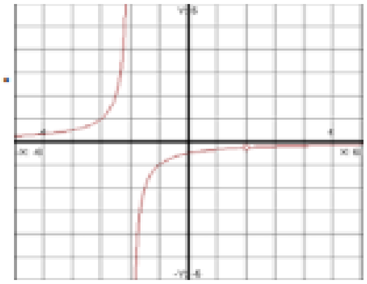
Example: Find the limit: $\lim\limits_{x \to 4^-} \sqrt{16 - x^2}$

We can still put in a 4 for x to get: $\sqrt{16 - 4^2} = 0$ Look at the graph to the right. As we approach 4 from the right, The y-value is approaching 0.



Example : Find the limit : $\lim\limits_{x \to 2^+} \frac{4-x}{x^2-4}$

$\lim\limits_{x \to 2^+} \frac{2-x}{(x-2)(x+2)} \to \lim\limits_{x \to 2^+} \frac{-(-2+x)}{(x-2)(x+2)} = \lim\limits_{x \to 2^+} \frac{-1}{(x+2)} =$



Example : - Find the limit : $\lim\limits_{x \to 1^-} (\frac{1}{x+1})(\frac{x+0}{x})(\frac{5-x}{7})$

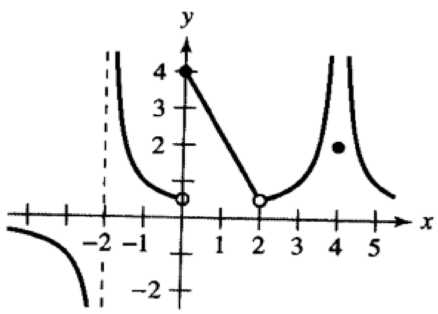
Given a function f(x) if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Then the normal limit is exist

Example:-

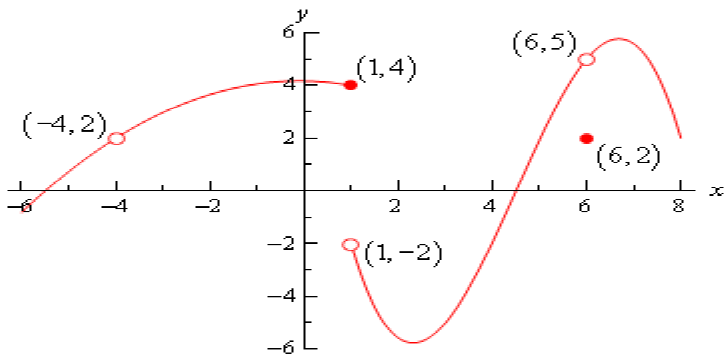
Use the graph of f(x) below to find the following:
$f(0), f(2), f(-2), \lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$,$\lim_{x\to 2} f(x)$ $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$, $\lim_{x\to -2} f(x)$.



$f(0) = 4$	Notice you are finding the y-value when x = 0. A closed circle is here the graph is defined
$f(2)$ =Undefined	For this one, there is no closed circle at the x = 2. So, nothing is defined here.
$f(-2)$ =Undefined	There is no closed circles here either. We have a vertical asymptote, so nothing will be defined here
$\lim_{x\to 2^+} f(x) = \frac{1}{2}$	You are seeing what the y-value is approaching as x approaches 2 from the right.
$\lim_{x\to 2^-} f(x) = \frac{1}{2}$	You are seeing what the y-value is approaching as x approaches 2 from the left.
$\lim_{x\to 2} f(x) = \frac{1}{2}$	Since the limit from the left and right are the same then our limit exists and is also equal to 1.
$\lim_{x\to 0^+} f(x) = 4$	You are seeing what the y-value is approaching as x approaches 0 from the right.
$\lim_{x\to 0^-} f(x) = \frac{1}{2}$	You are seeing what the y-value is approaching as x approaches 0 from the left.
$\lim_{x\to 0} f(x) = do\ not\ exist$	Since the limit from the left and from the right are not the same, the limit does not exist.

$\lim\limits_{x \to -2} f(x) = do\ not\ exist$ Since the limit from the left and from the right are not the same, the limit does not exist.

Example: Given the following graph,



Solution

compute each of the following.

(a) $f(-4)$ **(b)** $\lim\limits_{x \to -4^-} f(x)$ **(c)** $\lim\limits_{x \to -4^+} f(x)$ **(d)** $\lim\limits_{x \to -4} f(x)$

(e) $f(1)$ **(f)** $\lim\limits_{x \to 1^-} f(x)$ **(g)** $\lim\limits_{x \to 1^+} f(x)$ **(h)** $\lim\limits_{x \to 1} f(x)$

(i) $f(6)$ **(j)** $\lim\limits_{x \to 6^-} f(x)$ **(k)** $\lim\limits_{x \to 6^+} f(x)$ **(l)** $\lim\limits_{x \to 6} f(x)$

(a) $\boldsymbol{f(-4)}$ Doesn't exist. There is no closed dot for this value of *x* and so the function Doesn't exist at this point.

(b) $\boldsymbol{\lim\limits_{x \to 4^+} f(x) = 2}$ The function is approaching a value of 2 as *x* moves in towards -4 from the left.

(c) $\boldsymbol{\lim\limits_{x \to -4^+} f(x) = 2}$ The function is approaching a value of 2 as *x* moves in towards -4 from the right.

(d) $\boldsymbol{\lim\limits_{x \to -4} f(x) = 2}$ We can do this one of two ways. Either we can use the fact here and notice that the two one-sided limits are the same and so the normal limit must exist and have the same value as the one-sided limits or just get the answer from the graph. Also recall that a limit can exist at a point even if the function doesn't exist at that point.

(e) $f(1) = 4$The function will take on the *y* value where the closed dot is.

(f) $\boldsymbol{\lim\limits_{x \to -} f(x) = 4}$ The function is approaching a value of 4 as *x* moves in towards 1 from the left.

(g) $\lim\limits_{x\to1^+} f(x) = -2$ The function is approaching a value of -2 as x moves in towards 1 from the right. Remember that the limit does NOT care about what the function is actually doing at the point, it only cares about what the function is doing around the point. In this case, always staying to the right of x=1, the function is approaching a value of -2 and so the limit is -2. The limit is not 4, as that is value of the function at the point and again the limit doesn't care about that.

(h) $\lim\limits_{x\to1} f(x)$ doesn't exist. The two one-sided limits both exist, however they are different and so the normal limit doesn't exist.

(i) $f(6) = 2$ The function will take on the *y* value where the closed dot is.

(j) $\lim\limits_{x\to6^-} f(x) = 5$ The function is approaching a value of 5 as x moves in towards 6 from the left.

(k) $\lim\limits_{x\to6^+} f(x) = 5$ The function is approaching a value of 5 as x moves in towards 6 from the right.

(l) $\lim\limits_{x\to6} f(x) = 5$ Again, we can use either the graph or the fact to get this. Also, once more remember that the limit doesn't care what is happening at the point and so it's possible for the limit to have a different value than the function at a point. When dealing with limits we've always got to remember that limits simply do not care about what the function is doing at the point in question. Limits are only concerned with what the function is doing around the point.

Continuity and Discontinuity

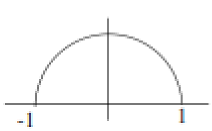
1- One-sided limits

We begin by expanding the notion of limit to include what are called one-sided limits, where x approaches a only from one side — the right or the left. The terminology and notation is:.

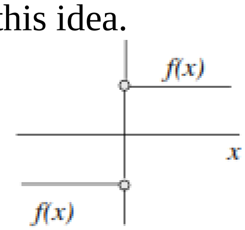
Right-hand limit $\lim\limits_{x\to a^+} f(x)$ **(x comes from the right, x > a)**

Left-hand limit $\lim\limits_{x\to a^-} f(x)$ **(x comes from the left, x < a)**

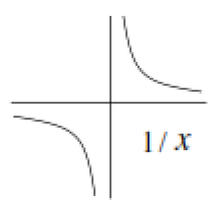
Since we use limits informally, a few examples will be enough to indicate the usefulness of this idea.



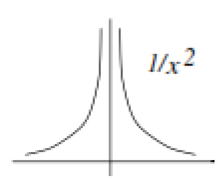
Ex. 1



Ex.2



Ex. 3



Ex.4

(As the picture shows , at the two endpoints of the domain , we only have a one – sided limit)

Example 2. Set $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

$$Then \lim_{x\to 0^-} f(x) = -1 , \lim_{x\to 0^+} f(x) = 1$$

Example 3 $\lim_{x\to 0^+} \frac{1}{x} = \infty$ $\qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$

Example 4 $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$ $\qquad \lim_{x\to 0^-} \frac{1}{x^2} = \infty$

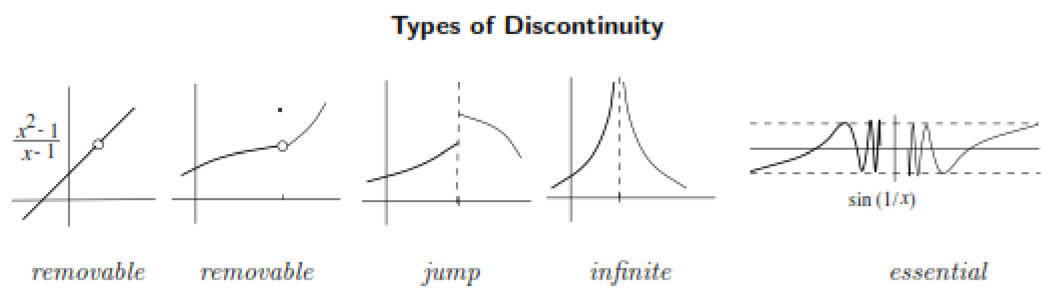
The relationship between the one-sided limits and the usual (two-sided) limit is given by

$$\lim_{x\to a} f(x) = L \leftrightarrow \lim_{a\to a^-} f(x) = L \ and \ lim_{a\to a^+} f(x) = L \ and$$

In words, the (two-sided) limit exists if and only if both one-sided limits exist and are equal. This shows for example that in Examples 2 and 3 above, $\lim_{x\to 0} f(x)$does not exist.

2. Continuity

$\boldsymbol{f(x)}$ is continuous at <u>**a**</u> if $\boldsymbol{\lim_{x\to a} f(x) = f(a)}$ We say a function is continuous on an interval [a, b] if it is defined on that interval and continuous at every point of that interval.



In a removable discontinuity, $\lim_{x\to a} f(x)$ exists, but $\lim_{x\to a} f(x) \neq f(a)$. This may be because $f(a)$ is undefined, or because $f(a)$ has the "wrong" value. The discontinuity can be removed by changing the definition of $f(x)$ at a so that its new value there is $\lim_{x\to a} f(x)$In the left – most picture, $\frac{x^2-1}{x-1}$ is undefined when x = 1, but if the definition of the function is completed by setting $f(1) = 2$, it becomes continuous — the hole in its graph is "filled in".

In a jump discontinuity , the right – and left – hand limits both exist , but are not equal . Thus , $\lim_{x \to a} f(x)$ does not exist .The size of the jump is the difference between the right – and left – hand limits .

In an infinite discontinuity (Examples 3 and 4), the one-sided limits exist (perhaps as ∞ or −∞), and at least one of them is ±∞.

An essential discontinuity is one which isn't of the three previous types — at least one of the one-sided limits doesn't exist (not even as ±∞). Though sin (1/x) is a standard simple example of a function with an essential discontinuity at 0, in applications they arise rarely, presumably because Mother Nature has no use for them.

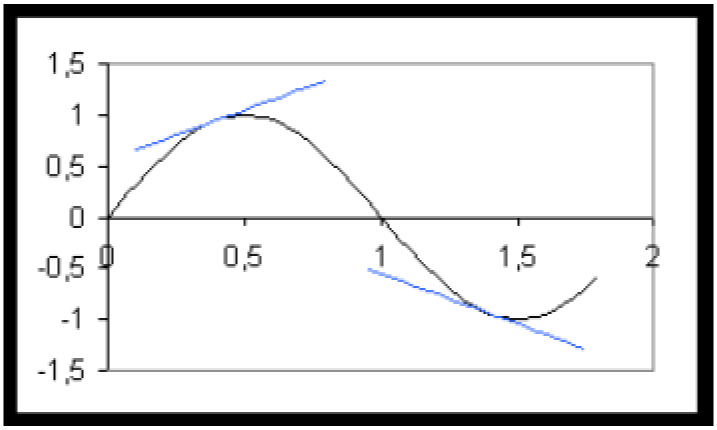
Definition of derivative

The derivative of a function *f* at a point, written, is given by:

$$\boldsymbol{f\acute{}(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

if this limit exists.

$f(x)'$ represents the derivative of the function evaluated at point , the derivative of a function corresponds to **the slope its tangent line at one specific point.** The following illustration allows us to visualize the tangent line (in blue) of a given function at two distinct points. N ote that the slope of the tangent line varies from one point to the next. The value of the derivative of a function therefore depends on the po int in which we decide to evaluate it.



Basic derivation rules

1. **Constant multiples** Let

K be real constant and f(x) any given function .Then

$$\left(kf(x)\right)' = kf'(x)$$

Example :-

$1-(-5\,e^{x})' = -5e^{x}$

$2-(12lnx)' = 12(\frac{1}{x})$

2- Addition and subtraction of

functions Let f(x) and g(x) be two functions . Then

$$(\left(f(x) \pm g(x)\right)' = f'(x) \mp g'(x)$$

Example :-

$$\left(ln3x - \frac{1}{x^2} + 8\right) = (ln3x)' - (x^{-2})' + (8)'$$

$$= \frac{1}{x} + \frac{2}{x^3}$$

3. Product rule

Let f(x) and g(x) be two functions. Then the derivate of the product

$$(\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)$$

Example:-

$$(3\sqrt{x}\,lnx)' = \left(\frac{3}{2}\right)x^{\frac{-1}{2}}lnx + 3x^{-\frac{1}{2}}$$

4.Quotient rule

Let f(x) and g(x) be two functions. Then the derivate of the product

$$\left(\frac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}}\right)' = \frac{\boldsymbol{f'(x)g(x) - f(x)g'(x)}}{\boldsymbol{(g(x))^2}}$$

Example :

$$\left(\frac{3\sqrt{x}}{lnx}\right)' = \frac{3x^{\frac{-1}{2}}(\,lnx-2)}{2(lnx)^2}$$

Evaluation of the slope of the tangent at one points

The derivative function $f(x)'$represents the slope of the tangent line at f(x) at all points x. we will often have to evaluate this slope at a specific point . To evaluate the slope of the tangent of the function f(x) at the point x=1 for example we most certainly cannot calculate f(1) and derive the value we would then obtain a slope of 0 since f(1) is a constant Instead, we need to find the derivative f(x)′ at all points and then evaluate it at x=1. We will use the notation f(a)′ to represent the derivative of the function evaluated at the point x=a

Example :-

Evaluate the slope of the function $f(x) = x^3e^x$ at point x= 0

Solution

We must first find the derivative at all points,

$$\boldsymbol{f(x)' = 3x^2e^x + x^3e^x}$$

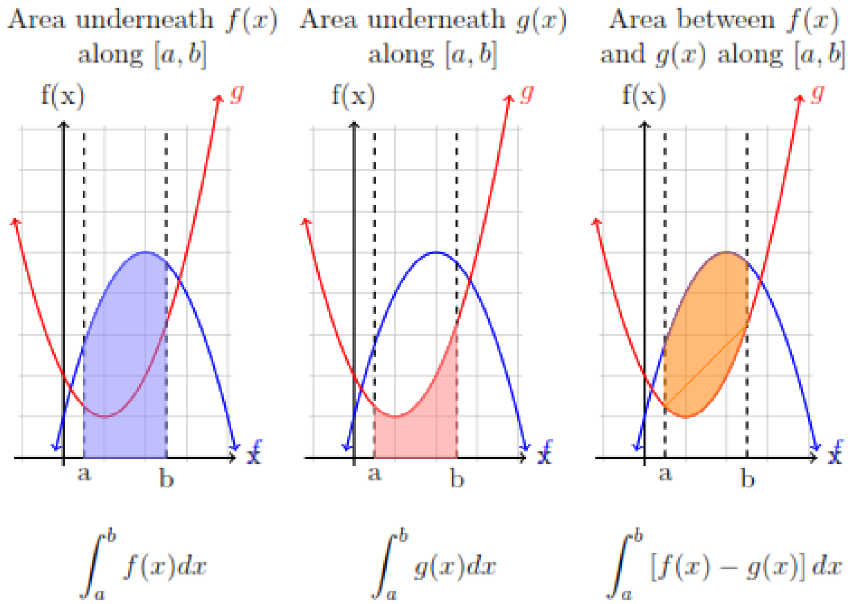
Evaluated at x=0, we obtain

$$\boldsymbol{f(0)' = 30^2e^0 + 0^3e^{x0} = 0}$$

The slope of the function $f(x) = x^3e^x$ therefore zero at x= 0.well will let you verify that this is not the case at points x =1.

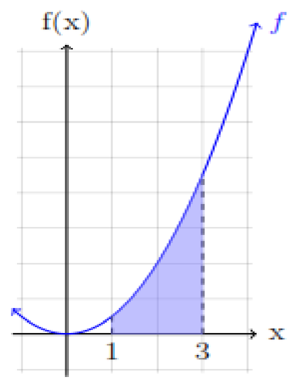
Using Integration to Find Areas

The definite integral $\int_a^b f(x)dx$ rep- resents the area between the function $\mathbf{f}(x)$ and the x-axis along the interval [a, b].



Example 1:- Find the area bounded by $f(x) = \frac{x^2}{2}$ and the x-axis along the interval [1; 3]

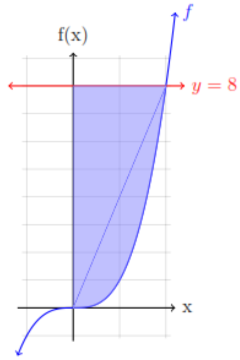
Solution.



$$\text{Area} = \int_1^3 \frac{x^2}{2} dx = \frac{x^3}{6} \Big|_1^3 = \frac{27}{6} - \frac{1}{6} = \frac{26}{6} = \frac{13}{2}$$

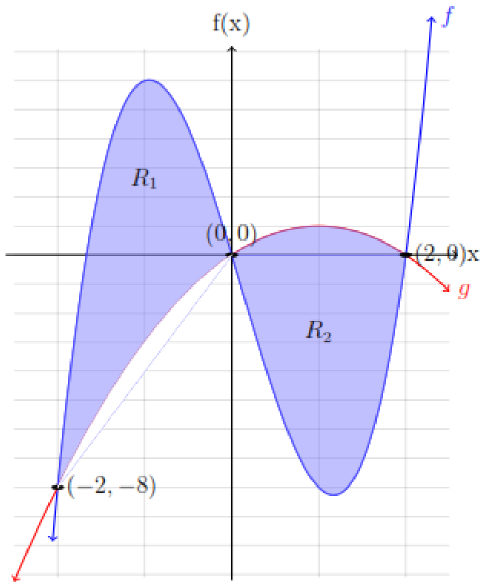
Example 2:- Find the area region bounded by positive y – axis y = 8 and $f(x) = x^3$ **Solution**.

We know that we are bounded by positive y – axis, which means we should begin from x =0, the second point by seeing where $f(x) = x^3$ intersects the horizontal line y = 8, $x^3 = 8 \rightarrow x = 2$ the interval [0, 2] .This gives



$$\text{Area} = \int_0^2 (8 - x^3)dx = 8x - \frac{x^4}{4} \Big|_0^2$$
$$= (16 - 4) - (0 - 0) = 12$$

Example 3:- Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.



Solution. Let’s find where the two functions intersect

$$3x^3 - x^2 - 10x = -x^2 + 2x$$
$$\rightarrow 3x^3 - 12x = 0$$
$$\rightarrow 3x(x^2 - 4) = 0 \rightarrow 3x(x + 2)(x - 2) = 0 \rightarrow x = -2, 0 \text{ and } 2$$

$$Area = Area(R1) + Area\ (R2)$$

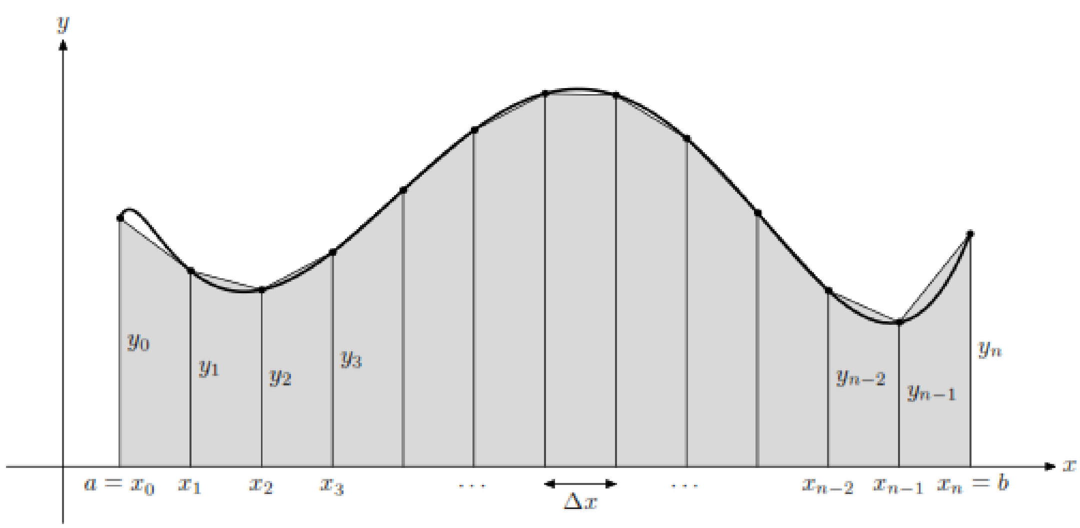
$$Area = \int_{-2}^{0} [(3x^3 - x^2 - 10x) - (-x^2 + 2x)]dx$$
$$+ \int_{0}^{2} [(-x^2 + 2x) - (3x^3 - x^2 \quad - 10x)dx$$

$=\int_{-2}^{0}[(3x^3 - 12x)dx \ + \int_0^2[12x - 3x^3]\,dx$
$=\left(\frac{3x^2}{4} - 6x^2|_{-2}^{0}\right) + \left(6x^2 - \frac{3x^4}{4}\ |_0^2\right) = [0 - (12 - 24)] + [(24 - 12) - 0\,] = 24$

Trapezoid Rule

A method for approximating a definite integral $\int_a^b f(x)dx$ using linear approximations of f. **<u>The trapezoids are drawn</u>** as shown below. The bases are vertical lines. To use the trapezoid rule follow these two steps:

1. Partition [a, b] into set {x_o , x_1, x_2 , x_3, -----------x_n} so that there are n sub – intervals of equal width .
2. The integral $\int_a^b f(x)dx$ is estimated by adding the areas of all the trapezoid as illustrated below .



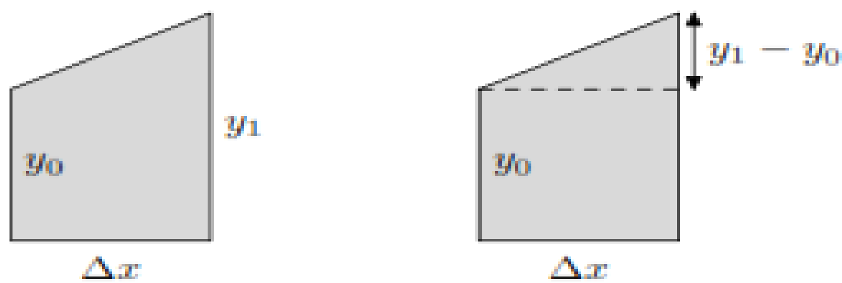
3. The width Δx of each sub – interval is given by $\boldsymbol{\Delta x = \frac{b-a}{n}}$ using the n+1

$$x_o = a\,, x_1 = a + \ \Delta x\,, \dots\dots, x_n = a + n\Delta x = b$$

We can compute the value of $f(x)$at these points .

$$y_o = f(x_0)\,, y_1 = f(x_1)\,, y_2 = f(x_2)\,, - - - - - -, y_n = f(x_n)\,,$$

The area of a trapezoid is obtained by adding the area of a rectangle and triangle .



$$A = y_0\Delta x + \frac{1}{2}(y_1 - y_0)\Delta x = \frac{(y_0 + y_1)\Delta x}{2}$$

$$\int_a^b f(x)dx = \frac{(y_0 + y_1)\Delta x}{2} + \frac{(y_1 + y_2)\Delta x}{2} + \frac{(y_2 + y_3)\Delta x}{2} + \cdots \frac{(y_{n-1} + y_n)\Delta x}{2}$$

$$\int_a^b f(x)dx = \frac{\Delta x}{2}\,[\,f(x_o) + 2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Which simplifies to the trapezoidal rule formula

$$\boxed{\int_a^b \boldsymbol{f(x)dx} = \frac{\boldsymbol{\Delta x}}{\mathbf{2}}\,[\,(\boldsymbol{y_o}) + \mathbf{2}(\boldsymbol{y_1}) + \mathbf{2}(\boldsymbol{y_3}) + \cdots + \mathbf{2}(\boldsymbol{y_{n-1}}) + (\boldsymbol{y_n})]}$$

Example: - Use the trapezoidal rule with n = 8 to estimate $\int_{\mathbf{1}}^{\mathbf{5}} \sqrt{\mathbf{1} + \boldsymbol{x^2}}$

Solution :- for n = 8 , we have $\boldsymbol{\Delta x} = \frac{\mathbf{5-1}}{\mathbf{8}} = \mathbf{0.5}$ **,**we compute the values of $\boldsymbol{y_o, y_1, y_2 \dots\dots y_8}$

x									
	x_0=a	$x_1 = a + \Delta x$	$x_2 = a + 2\Delta x$	$x_3 = a + 3\Delta x$	$x_4 = a + 4\Delta x$	$x_5 = a + 5\Delta x$	$x_6 = a + 6\Delta x$	$x_7 = a + 7\Delta x$	$x_8 = a + 8\Delta x$
	1	1.5	2	2.5	3	3.5	4	4.5	5
y=$\sqrt{1+x^2}$	y_o	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
	$\sqrt{2}$	$\sqrt{3.25}$	$\sqrt{5}$	$\sqrt{7.25}$	$\sqrt{10}$	$\sqrt{13.25}$	$\sqrt{17}$	$\sqrt{21.25}$	$\sqrt{26}$

Therefore,

$$\int_1^5 \sqrt{1+x^2} \approx$$

$\frac{0.5}{2}$ ($\sqrt{2}$+2$\sqrt{3.25}$ + 2$\sqrt{5}$ + 2$\sqrt{7.25}$ + 2$\sqrt{10}$ + 2$\sqrt{13.25}$ + 2$\sqrt{17}$ + 2$\sqrt{21.25}$ + $\sqrt{26}$
= 12.76

Example 2. The following points were found empirically

x	2.1	2.4	2.7	3.0	3.3	3.6
y	3.2	2.7	2.9	3.5	4.1	5.2

Use the trapezoidal rule to estimate $\int_{2.1}^{3.6} ydx$

Solution: - we see that Δx = 0.3 Therefore,

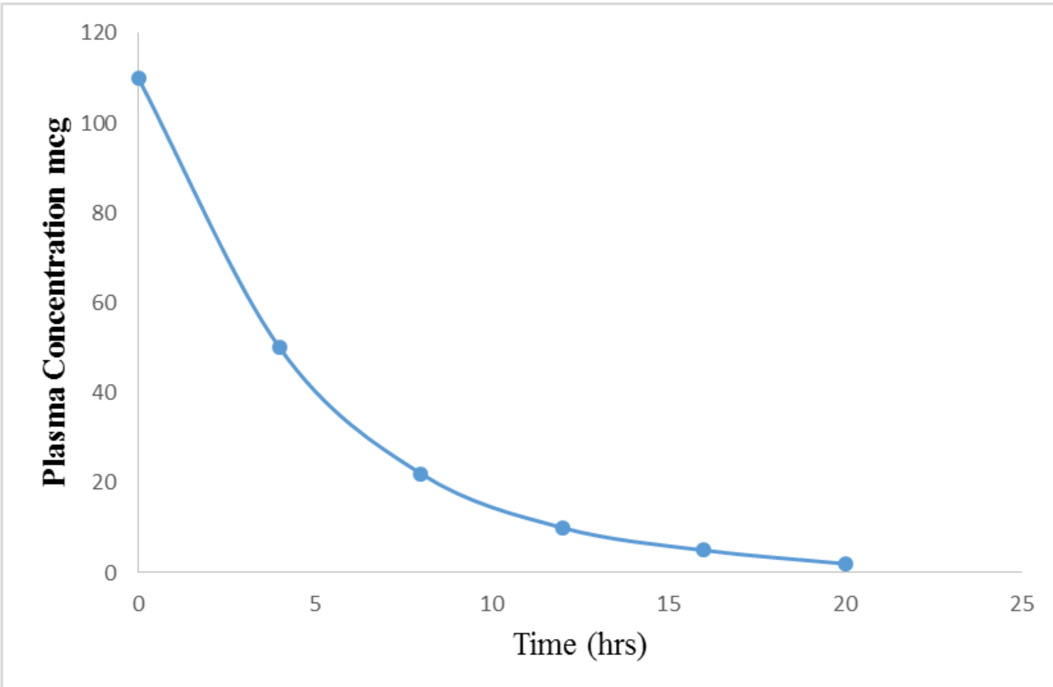
$\int_{2.1}^{3.6} \sqrt{ydx} \approx \frac{0.3}{2}$ (3.2+2 × 2.7 + 2 × 2.9 + 2 × 3.5 + 2 × 4.1 + 5.2
≈ 5.22

Calculation of AUC Using the Trapezoidal Rule (Pharmaceutical)

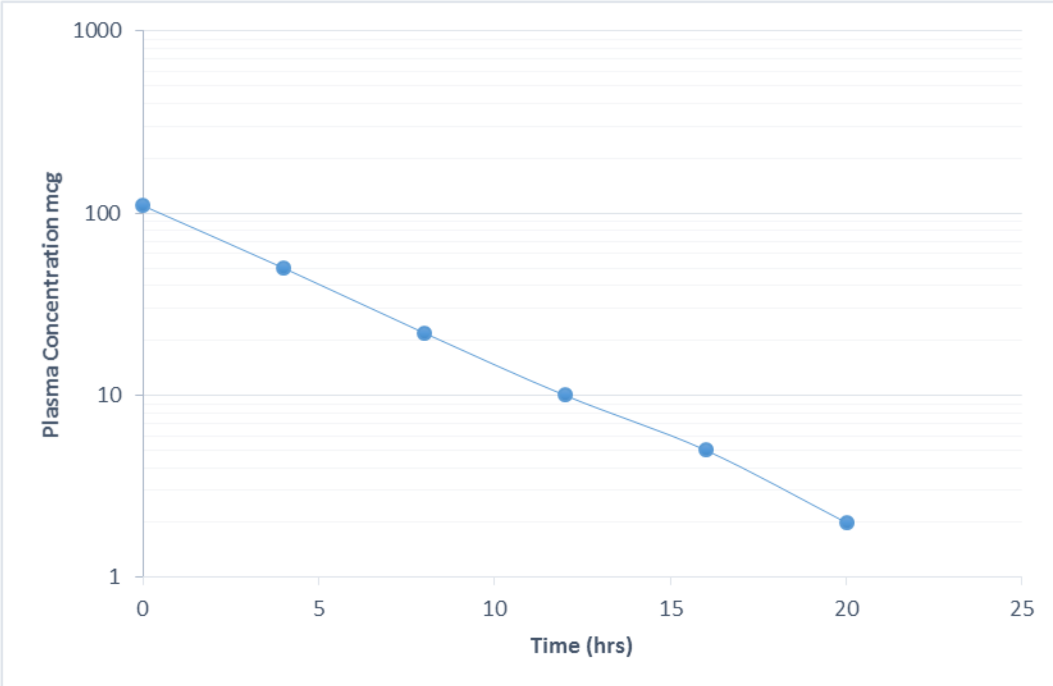
1. A 3 year old, 15 kg patient was brought in for surgery and was given a 110 mcg/kg iv bolus injection of a muscle relaxant. The plasma concentrations were measured post injection and noted in the table below:

Time (hrs)	Concentration (mcg/L)
0	110
4	50
8	22
12	10
16	5
20	2

Determine the following pharmacokinetic parameters of the drug for this patient:



Let's plot the data on a semi-log scale:



we get the concentration at 4 hrs which is 50 mcg/L and at 16 hrs which is 5 mcg/L.

$$k_e = \frac{\ln C_1 - \ln C_2}{t_2 - t_1} = \frac{\ln\left(50\,{}^{mcg}\!/_{L}\right) - \ln\left(5\,{}^{mcg}\!/_{L}\right)}{16hr - 4hr} = 0.2hr^{-1}$$

$$AUC_{t1\to t2} = \frac{c_{t2} + c_{t1}}{2} \times (t_2 - t_1)$$

$$AUC_{0h\to 4h} = \frac{\frac{50mcg}{L} + \frac{110mcg}{L}}{2} \times (4hr - 0hr) = 320mcg.\frac{hr}{L}$$

$$AUC_{4h\to 8h} = 144mcg.\frac{hr}{L}$$

$$AUC_{8h\to 12h} = 64mcg.\frac{hr}{L}$$

$$AUC_{12h\to 16h} = 30mcg.\frac{hr}{L}$$

$$AUC_{16h\to 20h} = 14mcg.\frac{hr}{L}$$

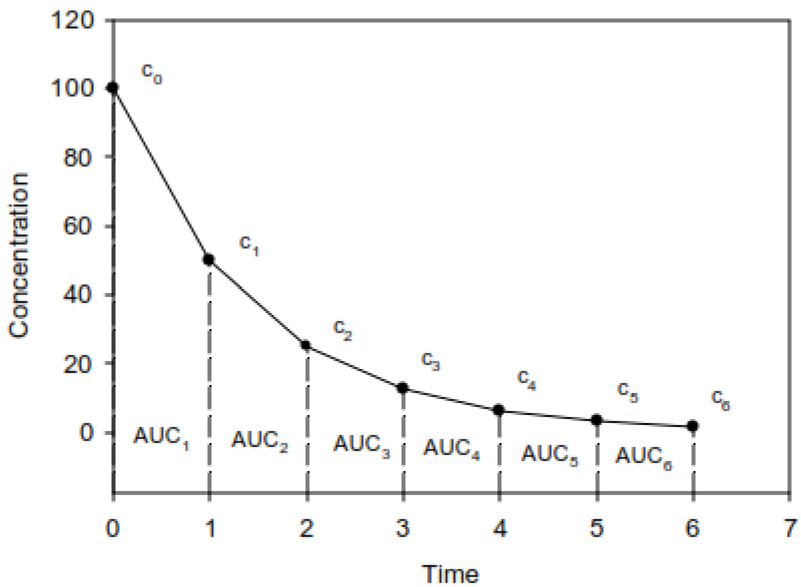
$$AUC_{20\to\infty} = \frac{C_{20h}}{k_e} = \frac{2mcg/L}{0.2hr^{-1}} = 10mcg.\frac{hr}{L}$$

$$AUC_{0\to\infty} = AUC_{0h\to 4h} + AUC_{4h\to 8h} + AUC_{8h\to 12h} + AUC_{12h\to 16h} + AUC_{16h\to 20h} + AUC_{20\to\infty}$$

$$AUC_{0\to\infty} = (320 + 144 + 64 + 30 + 14 + 10)\frac{mcg.hr}{L} = 582\frac{mcg.hr}{L}$$

Consider the following concentration time data:

Time (hr)	0	1	2	3	4	5	6
Conc (mg/L)	100	50	25	12.5	6.25	3.13	1.56



Area of trapezoid with the first time interval:

$$AUC_1 = \frac{c_0 + c_1}{2} \times Time_{1-0}$$

$$AUC_1 = \frac{100 + 50}{2} \times 1hr = 75mg - hr / L$$

Time (hr.)	Conc (mg/L)	Time Interval (hr.)	Average Concentration (mg/L)	Area (mg-hr/L)
0	100	-	-	-
1	50	1	75	75
2	25	1	37.5	37.5
3	12.5	1	18.75	18.75
4	6.25	1	9.375	9.375
5	3.13	1	4.688	4.688
6	1.56	1	2.344	2.344
			AUC* =	147.657

For AUC $_{0\text{-}\infty}$ = AUC $_{c0\text{-}c6}$ + AUC $_{c6\text{-}\infty}$

$$AUC_{c6-\infty} = \frac{c_6}{k} = \frac{1.56}{0.693} = 2.25$$

For AUC $_{0\text{-}\infty}$ = 147.657 + 2.25 = 149.91 mg-hr/L